CHAPTER 9

Time-Varying Fields and Maxwell's Equations

The basic relationships of the electrostatic field and the steady magnetic field were obtained in the previous eight chapters, and we are now ready to discuss time-varying fields. The discussion will be short, for vector analysis and vector calculus should now be more familiar tools; some of the relationships are unchanged, and most of the relationships are changed only slightly.

Two new concepts will be introduced: the electric field produced by a changing magnetic field and the magnetic field produced by a changing electric field. The first of these concepts resulted from experimental research by Michael Faraday and the second from the theoretical efforts of James Clerk Maxwell.

Maxwell actually was inspired by Faraday's experimental work and by the mental picture provided through the "lines of force" that Faraday introduced in developing his theory of electricity and magnetism. He was 40 years younger than Faraday, but they knew each other during the five years Maxwell spent in London as a young professor, a few years after Faraday had retired. Maxwell's theory was developed subsequent to his holding this university position while he was working alone at his home in Scotland. It occupied him for five years between the ages of 35 and 40.

The four basic equations of electromagnetic theory presented in this chapter bear his name. ■

9.1 FARADAY'S LAW

After Oersted[1] demonstrated in 1820 that an electric current affected a compass needle, Faraday professed his belief that if a current could produce a magnetic field, then a magnetic field should be able to produce a current. The concept of the "field"

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[1] Hans Christian Oersted was professor of physics at the University of Copenhagen in Denmark.

was not available at that time, and Faraday's goal was to show that a current could be produced by "magnetism."

He worked on this problem intermittently over a period of 10 years, until he was finally successful in 1831.[2] He wound two separate windings on an iron toroid and placed a galvanometer in one circuit and a battery in the other. Upon closing the battery circuit, he noted a momentary deflection of the galvanometer; a similar deflection in the opposite direction occurred when the battery was disconnected. This, of course, was the first experiment he made involving a *changing* magnetic field, and he followed it with a demonstration that either a *moving* magnetic field or a moving coil could also produce a galvanometer deflection.

In terms of fields, we now say that a time-varying magnetic field produces an *electromotive force* (emf) that may establish a current in a suitable closed circuit. An electromotive force is merely a voltage that arises from conductors moving in a magnetic field or from changing magnetic fields, and we shall define it in this section. Faraday's law is customarily stated as

$$\text{emf} = -\frac{d\Phi}{dt} \text{ V} \tag{1}$$

Equation (1) implies a closed path, although not necessarily a closed conducting path; the closed path, for example, might include a capacitor, or it might be a purely imaginary line in space. The magnetic flux is that flux which passes through any and every surface whose perimeter is the closed path, and $d\Phi/dt$ is the time rate of change of this flux.

A nonzero value of $d\Phi/dt$ may result from any of the following situations:

1. A time-changing flux linking a stationary closed path
2. Relative motion between a steady flux and a closed path
3. A combination of the two

The minus sign is an indication that the emf is in such a direction as to produce a current whose flux, if added to the original flux, would reduce the magnitude of the emf. This statement that the induced voltage acts to produce an opposing flux is known as *Lenz's law*.[3]

If the closed path is that taken by an N-turn filamentary conductor, it is often sufficiently accurate to consider the turns as coincident and let

$$\text{emf} = -N\frac{d\Phi}{dt} \tag{2}$$

where Φ is now interpreted as the flux passing through any one of N coincident paths.

[2] Joseph Henry produced similar results at Albany Academy in New York at about the same time.

[3] Henri Frederic Emile Lenz was born in Germany but worked in Russia. He published his law in 1834.

We need to define emf as used in (1) or (2). The emf is obviously a scalar, and (perhaps not so obviously) a dimensional check shows that it is measured in volts. We define the emf as

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} \tag{3}$$

and note that it is the voltage about a specific *closed path*. If any part of the path is changed, generally the emf changes. The departure from static results is clearly shown by (3), for an electric field intensity resulting from a static charge distribution must lead to zero potential difference about a closed path. In electrostatics, the line integral leads to a potential difference; with time-varying fields, the result is an emf or a voltage.

Replacing Φ in (1) with the surface integral of **B**, we have

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{4}$$

where the fingers of our right hand indicate the direction of the closed path, and our thumb indicates the direction of $d\mathbf{S}$. A flux density **B** in the direction of $d\mathbf{S}$ and increasing with time thus produces an average value of **E** which is *opposite* to the positive direction about the closed path. The right-handed relationship between the surface integral and the closed line integral in (4) should always be kept in mind during flux integrations and emf determinations.

We will divide our investigation into two parts by first finding the contribution to the total emf made by a changing field within a stationary path (transformer emf), and then we will consider a moving path within a constant (motional, or generator, emf).

We first consider a stationary path. The magnetic flux is the only time-varying quantity on the right side of (4), and a partial derivative may be taken under the integral sign,

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{5}$$

Before we apply this simple result to an example, let us obtain the point form of this integral equation. Applying Stokes' theorem to the closed line integral, we have

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

where the surface integrals may be taken over identical surfaces. The surfaces are perfectly general and may be chosen as differentials,

$$(\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6}$$

This is one of Maxwell's four equations as written in differential, or point, form, the form in which they are most generally used. Equation (5) is the integral form of this equation and is equivalent to Faraday's law as applied to a fixed path. If **B** is not a function of time, (5) and (6) evidently reduce to the electrostatic equations

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0 \quad \text{(electrostatics)}$$

and

$$\nabla \times \mathbf{E} = 0 \quad \text{(electrostatics)}$$

As an example of the interpretation of (5) and (6), let us assume a simple magnetic field which increases exponentially with time within the cylindrical region $\rho < b$,

$$\mathbf{B} = B_0 e^{kt} \mathbf{a}_z \tag{7}$$

where $B_0 =$ constant. Choosing the circular path $\rho = a$, $a < b$, in the $z = 0$ plane, along which E_ϕ must be constant by symmetry, we then have from (5)

$$\text{emf} = 2\pi a E_\phi = -k B_0 e^{kt} \pi a^2$$

The emf around this closed path is $-k B_0 e^{kt} \pi a^2$. It is proportional to a^2 because the magnetic flux density is uniform and the flux passing through the surface at any instant is proportional to the area.

If we now replace a with ρ, $\rho < b$, the electric field intensity at any point is

$$\mathbf{E} = -\tfrac{1}{2} k B_0 e^{kt} \rho \mathbf{a}_\phi \tag{8}$$

Let us now attempt to obtain the same answer from (6), which becomes

$$(\nabla \times \mathbf{E})_z = -k B_0 e^{kt} = \frac{1}{\rho} \frac{\partial(\rho E_\phi)}{\partial \rho}$$

Multiplying by ρ and integrating from 0 to ρ (treating t as a constant, since the derivative is a partial derivative),

$$-\tfrac{1}{2} k B_0 e^{kt} \rho^2 = \rho E_\phi$$

or

$$\mathbf{E} = -\tfrac{1}{2} k B_0 e^{kt} \rho \mathbf{a}_\phi$$

once again.

If B_0 is considered positive, a filamentary conductor of resistance R would have a current flowing in the negative $\mathbf{a}_\phi$ direction, and this current would establish a flux within the circular loop in the negative $\mathbf{a}_z$ direction. Because E_ϕ increases exponentially with time, the current and flux do also, and thus they tend to reduce the time rate of increase of the applied flux and the resultant emf in accordance with Lenz's law.

Before leaving this example, it is well to point out that the given field **B** does not satisfy all of Maxwell's equations. Such fields are often assumed (*always* in ac-circuit problems) and cause no difficulty when they are interpreted properly. They

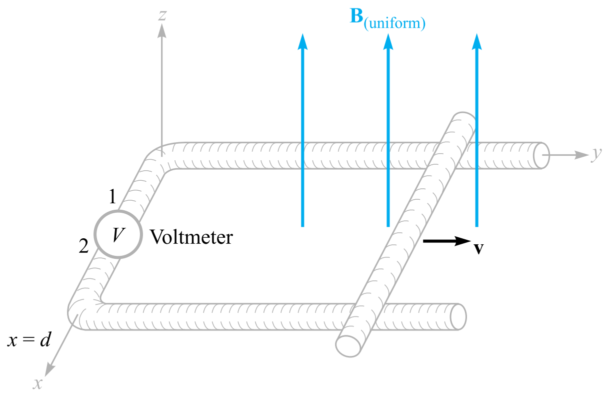

Figure 9.1 An example illustrating the application of Faraday's law to the case of a constant magnetic flux density **B** and a moving path. The shorting bar moves to the right with a velocity **v**, and the circuit is completed through the two rails and an extremely small high-resistance voltmeter. The voltmeter reading is $V_{12} = -Bvd$.

occasionally cause surprise, however. This particular field is discussed further in Problem 9.19 at the end of the chapter.

Now let us consider the case of a time-constant flux and a moving closed path. Before we derive any special results from Faraday's law (1), let us use the basic law to analyze the specific problem outlined in Figure 9.1. The closed circuit consists of two parallel conductors which are connected at one end by a high-resistance voltmeter of negligible dimensions and at the other end by a sliding bar moving at a velocity **v**. The magnetic flux density **B** is constant (in space and time) and is normal to the plane containing the closed path.

Let the position of the shorting bar be given by y; the flux passing through the surface within the closed path at any time t is then

$$\Phi = Byd$$

From (1), we obtain

$$\text{emf} = -\frac{d\Phi}{dt} = -B\frac{dy}{dt}d = -B\nu d \tag{9}$$

The emf is defined as $\oint \mathbf{E} \cdot d\mathbf{L}$ and we have a conducting path, so we may actually determine **E** at every point along the closed path. We found in electrostatics that the tangential component of **E** is zero at the surface of a conductor, and we shall show in Section 9.4 that the tangential component is zero at the surface of a *perfect* conductor ($\sigma = \infty$) for all time-varying conditions. This is equivalent to saying that a perfect conductor is a "short circuit." The entire closed path in Figure 9.1 may be considered a perfect conductor, with the exception of the voltmeter. The actual computation of $\oint \mathbf{E} \cdot d\mathbf{L}$ then must involve no contribution along the entire moving bar, both rails, and the voltmeter leads. Because we are integrating in a counterclockwise direction (keeping the interior of the positive side of the surface on our left as usual), the contribution $E\,\Delta L$ across the voltmeter must be $-Bvd$, showing that the electric field intensity in the instrument is directed from terminal 2 to terminal 1. For an up-scale reading, the positive terminal of the voltmeter should therefore be terminal 2.

The direction of the resultant small current flow may be confirmed by noting that the enclosed flux is reduced by a clockwise current in accordance with Lenz's law. The voltmeter terminal 2 is again seen to be the positive terminal.

Let us now consider this example using the concept of *motional emf*. The force on a charge Q moving at a velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is

$$\boxed{\mathbf{F} = Q\mathbf{v} \times \mathbf{B}}$$

or

$$\frac{\mathbf{F}}{Q} = \mathbf{v} \times \mathbf{B} \tag{10}$$

The sliding conducting bar is composed of positive and negative charges, and each experiences this force. The force per unit charge, as given by (10), is called the *motional* electric field intensity $\mathbf{E}_m$,

$$\boxed{\mathbf{E}_m = \mathbf{v} \times \mathbf{B}} \tag{11}$$

If the moving conductor were lifted off the rails, this electric field intensity would force electrons to one end of the bar (the far end) until the *static field* due to these charges just balanced the field induced by the motion of the bar. The resultant tangential electric field intensity would then be zero along the length of the bar.

The motional emf produced by the moving conductor is then

$$\text{emf} = \oint \mathbf{E}_m \cdot d\mathbf{L} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L} \tag{12}$$

where the last integral may have a nonzero value only along that portion of the path which is in motion, or along which $\mathbf{v}$ has some nonzero value. Evaluating the right side of (12), we obtain

$$\oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L} = \int_d^0 vB\,dx = -Bvd$$

as before. This is the total emf, since $\mathbf{B}$ is not a function of time.

In the case of a conductor moving in a uniform constant magnetic field, we may therefore ascribe a motional electric field intensity $\mathbf{E}_m = \mathbf{v} \times \mathbf{B}$ to every portion of the moving conductor and evaluate the resultant emf by

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = \oint \mathbf{E}_m \cdot d\mathbf{L} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L} \tag{13}$$

If the magnetic flux density is also changing with time, then we must include both contributions, the transformer emf (5) and the motional emf (12),

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L} \tag{14}$$

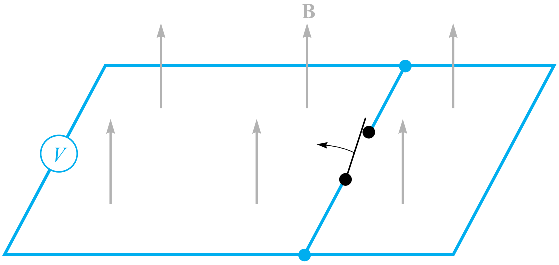

Figure 9.2 An apparent increase in flux linkages does not lead to an induced voltage when one part of a circuit is simply substituted for another by opening the switch. No indication will be observed on the voltmeter.

This expression is equivalent to the simple statement

$$\text{emf} = -\frac{d\Phi}{dt} \tag{1}$$

and either can be used to determine these induced voltages.

Although (1) appears simple, there are a few contrived examples in which its proper application is quite difficult. These usually involve sliding contacts or switches; they always involve the substitution of one part of a circuit by a new part.[4] As an example, consider the simple circuit of Figure 9.2, which contains several perfectly conducting wires, an ideal voltmeter, a uniform constant field **B**, and a switch. When the switch is opened, there is obviously more flux enclosed in the voltmeter circuit; however, it continues to read zero. The change in flux has not been produced by either a time-changing **B** [first term of (14)] or a conductor moving through a magnetic field [second part of (14)]. Instead, a new circuit has been substituted for the old. Thus it is necessary to use care in evaluating the change in flux linkages.

The separation of the emf into the two parts indicated by (14), one due to the time rate of change of **B** and the other to the motion of the circuit, is somewhat arbitrary in that it depends on the relative velocity of the *observer* and the system. A field that is changing with both time and space may look constant to an observer moving with the field. This line of reasoning is developed more fully in applying the special theory of relativity to electromagnetic theory.[5]

D9.1. Within a certain region, $\epsilon = 10^{-11}$ F/m and $\mu = 10^{-5}$ H/m. If $B_x = 2 \times 10^{-4} \cos 10^5 t \sin 10^{-3} y$ T: (*a*) use $\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$ to find **E**; (*b*) find the total magnetic flux passing through the surface $x = 0$, $0 < y < 40$ m, $0 < z < 2$ m,

[4] See Bewley, in References at the end of the chapter, particularly pp. 12–19.

[5] This is discussed in several of the references listed in the References at the end of the chapter. See Panofsky and Phillips, pp. 142–51; Owen, pp. 231–45; and Harman in several places.

at $t = 1\ \mu$s; (*c*) find the value of the closed line integral of **E** around the perimeter of the given surface.

Ans. $-20\,000 \sin 10^5 t \cos 10^{-3} y \mathbf{a}_z$ V/m; 0.318 mWb; −3.19 V

D9.2. With reference to the sliding bar shown in Figure 9.1, let $d = 7$ cm, $\mathbf{B} = 0.3\mathbf{a}_z$ T, and $\mathbf{v} = 0.1\mathbf{a}_y e^{20y}$ m/s. Let $y = 0$ at $t = 0$. Find: (*a*) $v(t = 0)$; (*b*) $y(t = 0.1)$; (*c*) $v(t = 0.1)$; (*d*) V_{12} at $t = 0.1$.

Ans. 0.1 m/s; 1.12 cm; 0.125 m/s; −2.63 mV

9.2 DISPLACEMENT CURRENT

Faraday's experimental law has been used to obtain one of Maxwell's equations in differential form,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{15}$$

which shows us that a time-changing magnetic field produces an electric field. Remembering the definition of curl, we see that this electric field has the special property of circulation; its line integral about a general closed path is not zero. Now let us turn our attention to the time-changing electric field.

We should first look at the point form of Ampère's circuital law as it applies to steady magnetic fields,

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{16}$$

and show its inadequacy for time-varying conditions by taking the divergence of each side,

$$\nabla \cdot \nabla \times \mathbf{H} \equiv 0 = \nabla \cdot \mathbf{J}$$

The divergence of the curl is identically zero, so $\nabla \cdot \mathbf{J}$ is also zero. However, the equation of continuity,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

then shows us that (16) can be true only if $\partial \rho_v / \partial t = 0$. This is an unrealistic limitation, and (16) must be amended before we can accept it for time-varying fields. Suppose we add an unknown term **G** to (16),

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{G}$$

Again taking the divergence, we have

$$0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{G}$$

Thus

$$\nabla \cdot \mathbf{G} = \frac{\partial \rho_v}{\partial t}$$

Replacing ρ_v with $\nabla \cdot \mathbf{D}$,

$$\nabla \cdot \mathbf{G} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}$$

from which we obtain the simplest solution for **G**,

$$\mathbf{G} = \frac{\partial \mathbf{D}}{\partial t}$$

Ampère's circuital law in point form therefore becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{17}$$

Equation (17) has not been derived. It is merely a form we have obtained that does not disagree with the continuity equation. It is also consistent with all our other results, and we accept it as we did each experimental law and the equations derived from it. We are building a theory, and we have every right to our equations *until they are proved wrong*. This has not yet been done.

We now have a second one of Maxwell's equations and shall investigate its significance. The additional term $\partial \mathbf{D}/\partial t$ has the dimensions of current density, amperes per square meter. Because it results from a time-varying electric flux density (or displacement density), Maxwell termed it a *displacement current density*. We sometimes denote it by $\mathbf{J}_d$:

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d$$

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}$$

This is the third type of current density we have met. Conduction current density,

$$\mathbf{J} = \sigma \mathbf{E}$$

is the motion of charge (usually electrons) in a region of zero net charge density, and convection current density,

$$\mathbf{J} = \rho_v \mathbf{v}$$

is the motion of volume charge density. Both are represented by **J** in (17). Bound current density is, of course, included in **H**. In a nonconducting medium in which no volume charge density is present, $\mathbf{J} = 0$, and then

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \quad (\text{if } \mathbf{J} = 0) \tag{18}$$

Notice the symmetry between (18) and (15):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{15}$$

Again, the analogy between the intensity vectors **E** and **H** and the flux density vectors **D** and **B** is apparent. We cannot place too much faith in this analogy, however, for it fails when we investigate forces on particles. The force on a charge is related to **E**

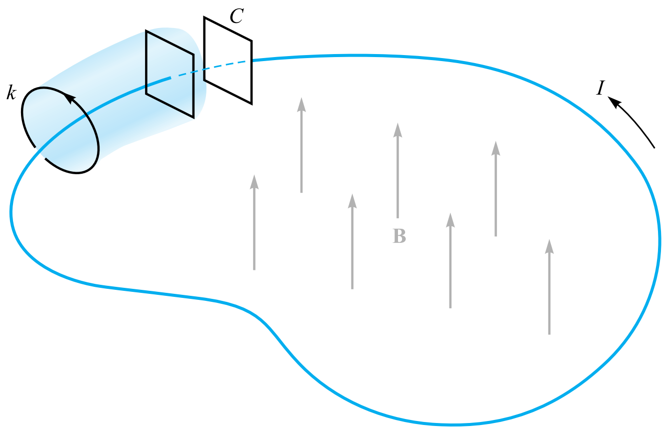

Figure 9.3 A filamentary conductor forms a loop connecting the two plates of a parallel-plate capacitor. A time-varying magnetic field inside the closed path produces an emf of $V_0 \cos \omega t$ around the closed path. The conduction current I is equal to the displacement current between the capacitor plates.

and to **B**, and some good arguments may be presented showing an analogy between **E** and **B** and between **D** and **H**. We omit them, however, and merely say that the concept of displacement current was probably suggested to Maxwell by the symmetry first mentioned in this paragraph.[6]

The total displacement current crossing any given surface is expressed by the surface integral,

$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{S} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

and we may obtain the time-varying version of Ampère's circuital law by integrating (17) over the surface S,

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

and applying Stokes' theorem,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I + I_d = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} \qquad (19)$$

What is the nature of displacement current density? Let us study the simple circuit of Figure 9.3, which contains a filamentary loop and a parallel-plate capacitor. Within

[6] The analogy that relates **B** to **D** and **H** to **E** is strongly advocated by Fano, Chu, and Adler (see References for Chapter 6); the case for comparing **B** to **E** and **D** to **H** is presented in Halliday and Resnick (see References for this chapter).

the loop, a magnetic field varying sinusoidally with time is applied to produce an emf about the closed path (the filament plus the dashed portion between the capacitor plates), which we shall take as

$$\text{emf} = V_0 \cos \omega t$$

Using elementary circuit theory and assuming that the loop has negligible resistance and inductance, we may obtain the current in the loop as

$$I = -\omega C V_0 \sin \omega t$$
$$= -\omega \frac{\epsilon S}{d} V_0 \sin \omega t$$

where the quantities ϵ, S, and d pertain to the capacitor. Let us apply Ampère's circuital law about the smaller closed circular path k and neglect displacement current for the moment:

$$\oint_k \mathbf{H} \cdot d\mathbf{L} = I_k$$

The path and the value of **H** along the path are both definite quantities (although difficult to determine), and $\oint_k \mathbf{H} \cdot d\mathbf{L}$ is a definite quantity. The current I_k is that current through every surface whose perimeter is the path k. If we choose a simple surface punctured by the filament, such as the plane circular surface defined by the circular path k, the current is evidently the conduction current. Suppose now we consider the closed path k as the mouth of a paper bag whose bottom passes between the capacitor plates. The bag is not pierced by the filament, and the conductor current is zero. Now we need to consider displacement current, for within the capacitor

$$D = \epsilon E = \epsilon \left(\frac{V_0}{d} \cos \omega t \right)$$

and therefore

$$I_d = \frac{\partial D}{\partial t} S = -\omega \frac{\epsilon S}{d} V_0 \sin \omega t$$

This is the same value as that of the conduction current in the filamentary loop. Therefore the application of Ampère's circuital law, including displacement current to the path k, leads to a definite value for the line integral of **H**. This value must be equal to the total current crossing the chosen surface. For some surfaces the current is almost entirely conduction current, but for those surfaces passing between the capacitor plates, the conduction current is zero, and it is the displacement current which is now equal to the closed line integral of **H**.

Physically, we should note that a capacitor stores charge and that the electric field between the capacitor plates is much greater than the small leakage fields outside. We therefore introduce little error when we neglect displacement current on all those surfaces which do not pass between the plates.

Displacement current is associated with time-varying electric fields and therefore exists in all imperfect conductors carrying a time-varying conduction current. The last part of the following drill problem indicates the reason why this additional current was never discovered experimentally.

D9.3. Find the amplitude of the displacement current density: (*a*) adjacent to an automobile antenna where the magnetic field intensity of an FM signal is $H_x = 0.15\cos[3.12(3\times 10^8 t - y)]$ A/m; (*b*) in the air space at a point within a large power distribution transformer where $\mathbf{B} = 0.8\cos[1.257\times 10^{-6}(3\times 10^8 t - x)]\mathbf{a}_y$ T; (*c*) within a large, oil-filled power capacitor where $\epsilon_r = 5$ and $\mathbf{E} = 0.9\cos[1.257 \times 10^{-6}(3\times 10^8 t - z\sqrt{5})]\mathbf{a}_x$ MV/m; (*d*) in a metallic conductor at 60 Hz, if $\epsilon = \epsilon_0$, $\mu = \mu_0$, $\sigma = 5.8\times 10^7$ S/m, and $\mathbf{J} = \sin(377t - 117.1z)\mathbf{a}_x$ MA/m^2.

Ans. 0.468 A/m^2; 0.800 A/m^2; 0.0150 A/m^2; 57.6 pA/m^2

9.3 MAXWELL'S EQUATIONS IN POINT FORM

We have already obtained two of Maxwell's equations for time-varying fields,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{20}$$

and

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{21}$$

The remaining two equations are unchanged from their non-time-varying form:

$$\nabla \cdot \mathbf{D} = \rho_v \tag{22}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{23}$$

Equation (22) essentially states that charge density is a source (or sink) of electric flux lines. Note that we can no longer say that *all* electric flux begins and terminates on charge, because the point form of Faraday's law (20) shows that **E**, and hence **D**, may have circulation if a changing magnetic field is present. Thus the lines of electric flux may form closed loops. However, the converse is still true, and every coulomb of charge must have one coulomb of electric flux diverging from it.

Equation (23) again acknowledges the fact that "magnetic charges," or poles, are not known to exist. Magnetic flux is always found in closed loops and never diverges from a point source.

These four equations form the basis of all electromagnetic theory. They are partial differential equations and relate the electric and magnetic fields to each other and to their sources, charge and current density. The auxiliary equations relating **D** and **E**,

$$\mathbf{D} = \epsilon \mathbf{E} \tag{24}$$

relating **B** and **H**,

$$\mathbf{B} = \mu \mathbf{H} \tag{25}$$

defining conduction current density,

$$\mathbf{J} = \sigma \mathbf{E} \tag{26}$$

and defining convection current density in terms of the volume charge density ρ_ν,

$$\mathbf{J} = \rho_\nu \mathbf{v} \tag{27}$$

are also required to define and relate the quantities appearing in Maxwell's equations.

The potentials V and **A** have not been included because they are not strictly necessary, although they are extremely useful. They will be discussed at the end of this chapter.

If we do not have "nice" materials to work with, then we should replace (24) and (25) with the relationships involving the polarization and magnetization fields,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{28}$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) \tag{29}$$

For linear materials we may relate **P** to **E**

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E} \tag{30}$$

and **M** to **H**

$$\mathbf{M} = \chi_m \mathbf{H} \tag{31}$$

Finally, because of its fundamental importance we should include the Lorentz force equation, written in point form as the force per unit volume,

$$\mathbf{f} = \rho_\nu (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{32}$$

The following chapters are devoted to the application of Maxwell's equations to several simple problems.

D9.4. Let $\mu = 10^{-5}$ H/m, $\epsilon = 4 \times 10^{-9}$ F/m, $\sigma = 0$, and $\rho_v = 0$. Find k (including units) so that each of the following pairs of fields satisfies Maxwell's equations: (*a*) $\mathbf{D} = 6\mathbf{a}_x - 2y\mathbf{a}_y + 2z\mathbf{a}_z$ nC/m^2, $\mathbf{H} = kx\mathbf{a}_x + 10y\mathbf{a}_y - 25z\mathbf{a}_z$ A/m; (*b*) $\mathbf{E} = (20y - kt)\mathbf{a}_x$ V/m, $\mathbf{H} = (y + 2 \times 10^6 t)\mathbf{a}_z$ A/m.

Ans. 15 A/m^2; -2.5×10^8 V/(m · s)

9.4 MAXWELL'S EQUATIONS IN INTEGRAL FORM

The integral forms of Maxwell's equations are usually easier to recognize in terms of the experimental laws from which they have been obtained by a generalization process. Experiments must treat physical macroscopic quantities, and their results therefore are expressed in terms of integral relationships. A differential equation always represents a theory. Let us now collect the integral forms of Maxwell's equations from Section 9.3.

Integrating (20) over a surface and applying Stokes' theorem, we obtain Faraday's law,

$$\oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{33}$$

and the same process applied to (21) yields Ampère's circuital law,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} \tag{34}$$

Gauss's laws for the electric and magnetic fields are obtained by integrating (22) and (23) throughout a volume and using the divergence theorem:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v dv \tag{35}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{36}$$

These four integral equations enable us to find the boundary conditions on **B**, **D**, **H**, and **E**, which are necessary to evaluate the constants obtained in solving Maxwell's equations in partial differential form. These boundary conditions are in general unchanged from their forms for static or steady fields, and the same methods may be used to obtain them. Between any two real physical media (where **K** must be zero on the boundary surface), (33) enables us to relate the tangential **E**-field components,

$$E_{t1} = E_{t2} \tag{37}$$

and from (34),

$$H_{t1} = H_{t2} \tag{38}$$

The surface integrals produce the boundary conditions on the normal components,

$$D_{N1} - D_{N2} = \rho_S \tag{39}$$

and

$$B_{N1} = B_{N2} \tag{40}$$

It is often desirable to idealize a physical problem by assuming a perfect conductor for which σ is infinite but $\mathbf{J}$ is finite. From Ohm's law, then, in a perfect conductor,

$$\mathbf{E} = 0$$

and it follows from the point form of Faraday's law that

$$\mathbf{H} = 0$$

for time-varying fields. The point form of Ampère's circuital law then shows that the finite value of $\mathbf{J}$ is

$$\mathbf{J} = 0$$

and current must be carried on the conductor surface as a surface current $\mathbf{K}$. Thus, if region 2 is a perfect conductor, (37) to (40) become, respectively,

$$E_{t1} = 0 \tag{41}$$

$$H_{t1} = K \quad (\mathbf{H}_{t1} = \mathbf{K} \times \mathbf{a}_N) \tag{42}$$

$$D_{N1} = \rho_s \tag{43}$$

$$B_{N1} = 0 \tag{44}$$

where $\mathbf{a}_N$ is an outward normal at the conductor surface.

Note that surface charge density is considered a physical possibility for either dielectrics, perfect conductors, or imperfect conductors, but that surface *current* density is assumed only in conjunction with perfect conductors.

The preceding boundary conditions are a very necessary part of Maxwell's equations. All real physical problems have boundaries and require the solution of Maxwell's equations in two or more regions and the matching of these solutions at the boundaries. In the case of perfect conductors, the solution of the equations within the conductor is trivial (all time-varying fields are zero), but the application of the boundary conditions (41) to (44) may be very difficult.

Certain fundamental properties of wave propagation are evident when Maxwell's equations are solved for an *unbounded* region. This problem is treated in Chapter 11. It represents the simplest application of Maxwell's equations because it is the only problem which does not require the application of any boundary conditions.

D9.5. The unit vector $0.64\mathbf{a}_x + 0.6\mathbf{a}_y - 0.48\mathbf{a}_z$ is directed from region 2 ($\epsilon_r = 2, \mu_r = 3, \sigma_2 = 0$) toward region 1 ($\epsilon_{r1} = 4, \mu_{r1} = 2, \sigma_1 = 0$). If $\mathbf{B}_1 = (\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z)\sin 300t$ T at point P in region 1 adjacent to the boundary, find the amplitude at P of: (*a*) $\mathbf{B}_{N1}$; (*b*) $\mathbf{B}_{t1}$; (*c*) $\mathbf{B}_{N2}$; (*d*) $\mathbf{B}_2$.

Ans. 2.00 T; 3.16 T; 2.00 T; 5.15 T

D9.6. The surface $y = 0$ is a perfectly conducting plane, whereas the region $y > 0$ has $\epsilon_r = 5$, $\mu_r = 3$, and $\sigma = 0$. Let $\mathbf{E} = 20\cos(2\times10^8 t - 2.58z)\mathbf{a}_y$ V/m for $y > 0$, and find at $t = 6$ ns; (*a*) ρ_S at $P(2, 0, 0.3)$; (*b*) $\mathbf{H}$ at P; (*c*) $\mathbf{K}$ at P.

Ans. 0.81 nC/m^2; $-62.3\mathbf{a}_x$ mA/m; $-62.3\mathbf{a}_z$ mA/m

9.5 THE RETARDED POTENTIALS

Animations

The time-varying potentials, usually called *retarded* potentials for a reason that we will see shortly, find their greatest application in radiation problems (to be addressed in Chapter 14) in which the distribution of the source is known approximately. We should remember that the scalar electric potential V may be expressed in terms of a static charge distribution,

$$V = \int_{\text{vol}} \frac{\rho_v dv}{4\pi\epsilon R} \quad \text{(static)} \tag{45}$$

and the vector magnetic potential may be found from a current distribution which is constant with time,

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu \mathbf{J}\, dv}{4\pi R} \quad \text{(dc)} \tag{46}$$

The differential equations satisfied by V,

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \quad \text{(static)} \tag{47}$$

and $\mathbf{A}$,

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \quad \text{(dc)} \tag{48}$$

may be regarded as the point forms of the integral equations (45) and (46), respectively.

Having found V and $\mathbf{A}$, the fundamental fields are then simply obtained by using the gradient,

$$\mathbf{E} = -\nabla V \quad \text{(static)} \tag{49}$$

or the curl,

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{(dc)} \tag{50}$$

We now wish to define suitable time-varying potentials which are consistent with the preceding expressions when only static charges and direct currents are involved.

Equation (50) apparently is still consistent with Maxwell's equations. These equations state that $\nabla \cdot \mathbf{B} = 0$, and the divergence of (50) leads to the divergence of the curl that is identically zero. Let us therefore tentatively accept (50) as satisfactory for time-varying fields and turn our attention to (49).

The inadequacy of (49) is obvious because application of the curl operation to each side and recognition of the curl of the gradient as being identically zero confront us with $\nabla \times \mathbf{E} = 0$. However, the point form of Faraday's law states that $\nabla \times \mathbf{E}$ is not generally zero, so let us try to effect an improvement by adding an unknown term to (49),

$$\mathbf{E} = -\nabla V + \mathbf{N}$$

taking the curl,

$$\nabla \times \mathbf{E} = 0 + \nabla \times \mathbf{N}$$

using the point form of Faraday's law,

$$\nabla \times \mathbf{N} = -\frac{\partial \mathbf{B}}{\partial t}$$

and using (50), giving us

$$\nabla \times \mathbf{N} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

or

$$\nabla \times \mathbf{N} = -\nabla \times \frac{\partial \mathbf{A}}{\partial t}$$

The simplest solution of this equation is

$$\mathbf{N} = -\frac{\partial \mathbf{A}}{\partial t}$$

and this leads to

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}} \tag{51}$$

We still must check (50) and (51) by substituting them into the remaining two of Maxwell's equations:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho_\nu$$

Doing this, we obtain the more complicated expressions

$$\frac{1}{\mu}\nabla \times \nabla \times \mathbf{A} = \mathbf{J} + \epsilon\left(-\nabla\frac{\partial V}{\partial t} - \frac{\partial^2 \mathbf{A}}{\partial t^2}\right)$$

and

$$\epsilon\left(-\nabla \cdot \nabla V - \frac{\partial}{\partial t}\nabla \cdot \mathbf{A}\right) = \rho_\nu$$

or

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \mu\epsilon\left(\nabla\frac{\partial V}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) \tag{52}$$

and

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho_\nu}{\epsilon} \tag{53}$$

There is no apparent inconsistency in (52) and (53). Under static or dc conditions $\nabla \cdot \mathbf{A} = 0$, and (52) and (53) reduce to (48) and (47), respectively. We will therefore assume that the time-varying potentials may be defined in such a way that **B** and **E** may be obtained from them through (50) and (51). These latter two equations do not serve, however, to define **A** and V *completely*. They represent necessary, but not sufficient, conditions. Our initial assumption was merely that $B = \nabla \times \mathbf{A}$, and a vector cannot be defined by giving its curl alone. Suppose, for example, that we have a very simple vector potential field in which A_y and A_z are zero. Expansion of (50) leads to

$$B_x = 0$$
$$B_y = \frac{\partial A_x}{\partial z}$$
$$B_z = -\frac{\partial A_x}{\partial y}$$

and we see that no information is available about the manner in which A_x varies with x. This information could be found if we also knew the value of the divergence of **A**, for in our example

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x}$$

Finally, we should note that our information about **A** is given only as partial derivatives and that a space-constant term might be added. In all physical problems in which the region of the solution extends to infinity, this constant term must be zero, for there can be no fields at infinity.

Generalizing from this simple example, we may say that a vector field is defined completely when both its curl and divergence are given and when its value is known at any one point (including infinity). We are therefore at liberty to specify the divergence of **A**, and we do so with an eye on (52) and (53), seeking the simplest expressions. We define

$$\nabla \cdot \mathbf{A} = -\mu\epsilon \frac{\partial V}{\partial t} \tag{54}$$

and (52) and (53) become

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} + \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{55}$$

and

$$\nabla^2 V = -\frac{\rho_\nu}{\epsilon} + \mu\epsilon \frac{\partial^2 V}{\partial t^2} \tag{56}$$

These equations are related to the wave equation, which will be discussed in Chapters 10 and 11. They show considerable symmetry, and we should be highly pleased with our definitions of V and $\mathbf{A}$,

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{50}$$

$$\nabla \cdot \mathbf{A} = -\mu\epsilon \frac{\partial V}{\partial t} \tag{54}$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{51}$$

The integral equivalents of (45) and (46) for the time-varying potentials follow from the definitions (50), (51), and (54), but we shall merely present the final results and indicate their general nature. In Chapter 11, we will find that any electromagnetic disturbance will travel at a velocity

$$\nu = \frac{1}{\sqrt{\mu\epsilon}}$$

through any homogeneous medium described by μ and ϵ. In the case of free space, this velocity turns out to be the velocity of light, approximately 3×10^8 m/s. It is logical, then, to suspect that the potential at any point is due not to the value of the charge density at some distant point at the same instant, but to its value at some previous time, because the effect propagates at a finite velocity. Thus (45) becomes

$$V = \int_{\text{vol}} \frac{[\rho_\nu]}{4\pi\epsilon R} d\nu \tag{57}$$

where $[\rho_\nu]$ indicates that every t appearing in the expression for ρ_ν has been replaced by a *retarded* time,

$$t' = t - \frac{R}{\nu}$$

Thus, if the charge density throughout space were given by

$$\rho_\nu = e^{-r} \cos \omega t$$

then

$$[\rho_\nu] = e^{-r} \cos \left[\omega \left(t - \frac{R}{\nu} \right) \right]$$

where R is the distance between the differential element of charge being considered and the point at which the potential is to be determined.

The retarded vector magnetic potential is given by

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu[\mathbf{J}]}{4\pi R} d\nu \tag{58}$$

The use of a retarded time has resulted in the time-varying potentials being given the name of retarded potentials. In Chapter 14 we will apply (58) to the simple situation of a differential current element in which I is a sinusoidal function of time. Other simple applications of (58) are considered in several problems at the end of this chapter.

We may summarize the use of the potentials by stating that a knowledge of the distribution of ρ_ν and **J** throughout space theoretically enables us to determine V and **A** from (57) and (58). The electric and magnetic fields are then obtained by applying (50) and (51). If the charge and current distributions are unknown, or reasonable approximations cannot be made for them, these potentials usually offer no easier path toward the solution than does the direct application of Maxwell's equations.

D9.7. A point charge of $4\cos 10^8\pi t\ \mu$C is located at $P_+(0, 0, 1.5)$, whereas $-4\cos 10^8\pi t\ \mu$C is at $P_-(0, 0, -1.5)$, both in free space. Find V at $P(r = 450, \theta, \phi = 0)$ at $t = 15$ ns for $\theta =$: (*a*) 0°; (*b*) 90°; (*c*) 45°.

Ans. 159.8 V; 0; 143 V

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CHAPTER 9 PROBLEMS

WWW Quizzes

9.1 In Figure 9.4, let $B = 0.2\cos 120\pi t$ T, and assume that the conductor joining the two ends of the resistor is perfect. It may be assumed that the magnetic field produced by $I(t)$ is negligible. Find (*a*) $V_{ab}(t)$; (*b*) $I(t)$.

9.2 In the example described by Figure 9.1, replace the constant magnetic flux density by the time-varying quantity $\mathbf{B} = B_0 \sin\omega t\ \mathbf{a}_z$. Assume that **U** is constant and that the displacement y of the bar is zero at $t = 0$. Find the emf at any time, t.

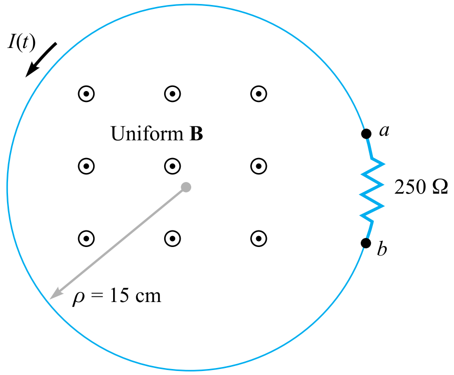

Figure 9.4 See Problem 9.1.

9.3 Given $\mathbf{H} = 300\mathbf{a}_z \cos(3 \times 10^8 t - y)$ A/m in free space, find the emf developed in the general $\mathbf{a}_\phi$ direction about the closed path having corners at (*a*) (0, 0, 0), (1, 0, 0), (1, 1, 0), and (0, 1, 0); (*b*) (0, 0, 0) $(2\pi, 0, 0)$, $(2\pi, 2\pi, 0)$, and $(0, 2\pi, 0)$.

9.4 A rectangular loop of wire containing a high-resistance voltmeter has corners initially at $(a/2, b/2, 0)$, $(-a/2, b/2, 0)$, $(-a/2, -b/2, 0)$, and $(a/2, -b/2, 0)$. The loop begins to rotate about the x axis at constant angular velocity ω, with the first-named corner moving in the $\mathbf{a}_z$ direction at $t = 0$. Assume a uniform magnetic flux density $\mathbf{B} = B_0\mathbf{a}_z$. Determine the induced emf in the rotating loop and specify the direction of the current.

9.5 The location of the sliding bar in Figure 9.5 is given by $x = 5t + 2t^3$, and the separation of the two rails is 20 cm. Let $\mathbf{B} = 0.8x^2\mathbf{a}_z$ T. Find the voltmeter reading at (*a*) $t = 0.4$ s; (*b*) $x = 0.6$ m.

9.6 Let the wire loop of Problem 9.4 be stationary in its $t = 0$ position and find the induced emf that results from a magnetic flux density given by $\mathbf{B}(y, t) = B_0 \cos(\omega t - \beta y)\,\mathbf{a}_z$, where ω and β are constants.

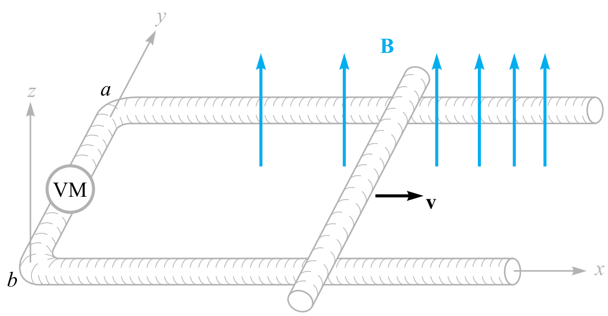

Figure 9.5 See Problem 9.5.

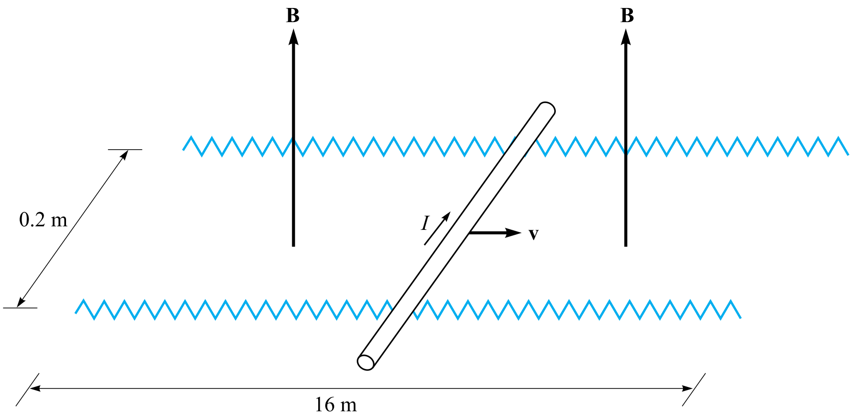

Figure 9.6 See Problem 9.7.

9.7 The rails in Figure 9.6 each have a resistance of 2.2 Ω/m. The bar moves to the right at a constant speed of 9 m/s in a uniform magnetic field of 0.8 T. Find $I(t)$, $0 < t < 1$ s, if the bar is at $x = 2$ m at $t = 0$ and (*a*) a 0.3 Ω resistor is present across the left end with the right end open-circuited; (*b*) a 0.3 Ω resistor is present across each end.

9.8 A perfectly conducting filament is formed into a circular ring of radius a. At one point, a resistance R is inserted into the circuit, and at another a battery of voltage V_0 is inserted. Assume that the loop current itself produces negligible magnetic field. (*a*) Apply Faraday's law, Eq. (4), evaluating each side of the equation carefully and independently to show the equality; (*b*) repeat part *a*, assuming the battery is removed, the ring is closed again, and a linearly increasing **B** field is applied in a direction normal to the loop surface.

9.9 A square filamentary loop of wire is 25 cm on a side and has a resistance of 125 Ω per meter length. The loop lies in the $z = 0$ plane with its corners at (0, 0, 0), (0.25, 0, 0), (0.25, 0.25, 0), and (0, 0.25, 0) at $t = 0$. The loop is moving with a velocity $v_y = 50$ m/s in the field $B_z = 8\cos(1.5 \times 10^8 t - 0.5x)\,\mu$T. Develop a function of time that expresses the ohmic power being delivered to the loop.

9.10 (*a*) Show that the ratio of the amplitudes of the conduction current density and the displacement current density is $\sigma/\omega\epsilon$ for the applied field $E = E_m \cos\omega t$. Assume $\mu = \mu_0$. (*b*) What is the amplitude ratio if the applied field is $E = E_m e^{-t/\tau}$, where τ is real?

9.11 Let the internal dimensions of a coaxial capacitor be $a = 1.2$ cm, $b = 4$ cm, and $l = 40$ cm. The homogeneous material inside the capacitor has the parameters $\epsilon = 10^{-11}$ F/m, $\mu = 10^{-5}$ H/m, and $\sigma = 10^{-5}$ S/m. If the electric field intensity is $\mathbf{E} = (10^6/\rho)\cos 10^5 t\,\mathbf{a}_\rho$ V/m, find (*a*) $\mathbf{J}$; (*b*) the total conduction current I_c through the capacitor; (c) the total displacement current I_d through the capacitor; (d) the ratio of the amplitude of I_d to that of I_c, the quality factor of the capacitor.

9.12 Find the displacement current density associated with the magnetic field $\mathbf{H} = A_1 \sin(4x)\cos(\omega t - \beta z)\,\mathbf{a}_x + A_2\cos(4x)\sin(\omega t - \beta z)\,\mathbf{a}_z$.

9.13 Consider the region defined by $|x|$, $|y|$, and $|z| < 1$. Let $\epsilon_r = 5$, $\mu_r = 4$, and $\sigma = 0$. If $J_d = 20\cos(1.5 \times 10^8 t - bx)\mathbf{a}_y$ μA/m^2 (a) find $\mathbf{D}$ and $\mathbf{E}$; (b) use the point form of Faraday's law and an integration with respect to time to find $\mathbf{B}$ and $\mathbf{H}$; (c) use $\nabla \times \mathbf{H} = \mathbf{J}_d + \mathbf{J}$ to find $\mathbf{J}_d$. (d) What is the numerical value of b?

9.14 A voltage source $V_0 \sin\omega t$ is connected between two concentric conducting spheres, $r = a$ and $r = b$, $b > a$, where the region between them is a material for which $\epsilon = \epsilon_r\epsilon_0$, $\mu = \mu_0$, and $\sigma = 0$. Find the total displacement current through the dielectric and compare it with the source current as determined from the capacitance (Section 6.3) and circuit-analysis methods.

9.15 Let $\mu = 3 \times 10^{-5}$ H/m, $\epsilon = 1.2 \times 10^{-10}$ F/m, and $\sigma = 0$ everywhere. If $\mathbf{H} = 2\cos(10^{10}t - \beta x)\mathbf{a}_z$ A/m, use Maxwell's equations to obtain expressions for $\mathbf{B}$, $\mathbf{D}$, $\mathbf{E}$, and β.

9.16 Derive the continuity equation from Maxwell's equations.

9.17 The electric field intensity in the region $0 < x < 5$, $0 < y < \pi/12$, $0 < z < 0.06$ m in free space is given by $\mathbf{E} = C \sin 12y \sin az \cos 2 \times 10^{10} t\mathbf{a}_x$ V/m. Beginning with the $\nabla \times \mathbf{E}$ relationship, use Maxwell's equations to find a numerical value for a, if it is known that a is greater than zero.

9.18 The parallel-plate transmission line shown in Figure 9.7 has dimensions $b = 4$ cm and $d = 8$ mm, while the medium between the plates is characterized by $\mu_r = 1$, $\epsilon_r = 20$, and $\sigma = 0$. Neglect fields outside the dielectric. Given the field $\mathbf{H} = 5\cos(10^9 t - \beta z)\mathbf{a}_y$ A/m, use Maxwell's

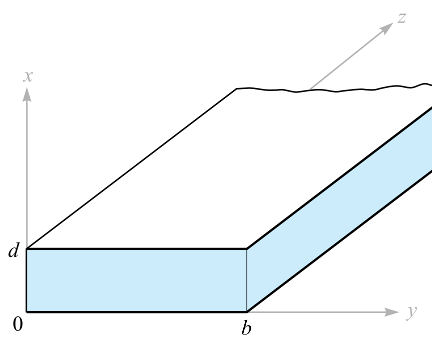

Figure 9.7 See Problem 9.18.

equations to help find (*a*) β, if $\beta > 0$; (*b*) the displacement current density at $z = 0$; (*c*) the total displacement current crossing the surface $x = 0.5d$, $0 < y < b$, $0 < z < 0.1$ m in the $\mathbf{a}_x$ direction.

9.19 In Section 9.1, Faraday's law was used to show that the field $\mathbf{E} = -\frac{1}{2}kB_0e^{kt}\rho\mathbf{a}_\phi$ results from the changing magnetic field $\mathbf{B} = B_0e^{kt}\mathbf{a}_z$. (*a*) Show that these fields do not satisfy Maxwell's other curl equation. (*b*) If we let $B_0 = 1$ T and $k = 10^6\ s^{-1}$, we are establishing a fairly large magnetic flux density in 1 μs. Use the $\nabla \times \mathbf{H}$ equation to show that the rate at which B_z should (but does not) change with ρ is only about 5×10^{-6} T per meter in free space at $t = 0$.

9.20 Given Maxwell's equations in point form, assume that all fields vary as e^{st} and write the equations without explicitly involving time.

9.21 (*a*) Show that under static field conditions, Eq. (55) reduces to Ampère's circuital law. (*b*) Verify that Eq. (51) becomes Faraday's law when we take the curl.

9.22 In a sourceless medium in which $\mathbf{J} = 0$ and $\rho_v = 0$, assume a rectangular coordinate system in which $\mathbf{E}$ and $\mathbf{H}$ are functions only of z and t. The medium has permittivity ϵ and permeability μ. (*a*) If $\mathbf{E} = E_x\mathbf{a}_x$ and $\mathbf{H} = H_y\mathbf{a}_y$, begin with Maxwell's equations and determine the second-order partial differential equation that E_x must satisfy. (*b*) Show that $E_x = E_0\cos(\omega t - \beta z)$ is a solution of that equation for a particular value of β. (*c*) Find β as a function of given parameters.

9.23 In region 1, $z < 0$, $\epsilon_1 = 2 \times 10^{-11}$ F/m, $\mu_1 = 2 \times 10^{-6}$ H/m, and $\sigma_1 = 4 \times 10^{-3}$ S/m; in region 2, $z > 0$, $\epsilon_2 = \epsilon_1/2$, $\mu_2 = 2\mu_1$, and $\sigma_2 = \sigma_1/4$. It is known that $\mathbf{E}_1 = (30\mathbf{a}_x + 20\mathbf{a}_y + 10\mathbf{a}_z)\cos 10^9 t$ V/m at $P(0, 0, 0^-)$. (*a*) Find $\mathbf{E}_{N1}$, $\mathbf{E}_{t1}$, $\mathbf{D}_{N1}$, and $\mathbf{D}_{t1}$ at P_1. (*b*) Find $\mathbf{J}_{N1}$ and $\mathbf{J}_{t1}$ at P_1. (*c*) Find $\mathbf{E}_{t2}$, $\mathbf{D}_{t2}$, and $\mathbf{J}_{t2}$ at $P_2(0, 0, 0^+)$. (*d*) (Harder) Use the continuity equation to help show that $J_{N1} - J_{N2} = \partial D_{N2}/\partial t - \partial D_{N1}/\partial t$, and then determine $\mathbf{D}_{N2}$, $\mathbf{J}_{N2}$, and $\mathbf{E}_{N2}$.

9.24 A vector potential is given as $\mathbf{A} = A_0\cos(\omega t - kz)\,\mathbf{a}_y$. (*a*) Assuming as many components as possible are zero, find $\mathbf{H}$, $\mathbf{E}$, and V. (*b*) Specify k in terms of A_0, ω, and the constants of the lossless medium, ϵ and μ.

9.25 In a region where $\mu_r = \epsilon_r = 1$ and $\sigma = 0$, the retarded potentials are given by $V = x(z - ct)$ V and $\mathbf{A} = x\left(\frac{z}{c} - t\right)\mathbf{a}_z$ Wb/m, where $c = 1\sqrt{\mu_0\epsilon_0}$. (*a*) Show that $\nabla \cdot \mathbf{A} = -\mu\epsilon\frac{\partial V}{\partial t}$. (*b*) Find $\mathbf{B}$, $\mathbf{H}$, $\mathbf{E}$, and $\mathbf{D}$. (*c*) Show that these results satisfy Maxwell's equations if $\mathbf{J}$ and ρ_v are zero.

9.26 Write Maxwell's equations in point form in terms of $\mathbf{E}$ and $\mathbf{H}$ as they apply to a sourceless medium, where $\mathbf{J}$ and ρ_v are both zero. Replace ϵ by μ, μ by ϵ, $\mathbf{E}$ by $\mathbf{H}$, and $\mathbf{H}$ by $-\mathbf{E}$, and show that the equations are unchanged. This is a more general expression of the *duality principle* in circuit theory.

CHAPTER 11

The Uniform Plane Wave

This chapter is concerned with the application of Maxwell's equations to the problem of electromagnetic wave propagation. The uniform plane wave represents the simplest case, and while it is appropriate for an introduction, it is of great practical importance. Waves encountered in practice can often be assumed to be of this form. In this study, we will explore the basic principles of electromagnetic wave propagation, and we will come to understand the physical processes that determine the speed of propagation and the extent to which attenuation may occur. We will derive and use the Poynting theorem to find the power carried by a wave. Finally, we will learn how to describe wave polarization. ■

11.1 WAVE PROPAGATION IN FREE SPACE

We begin with a quick study of Maxwell's equations, in which we look for clues of wave phenomena. In Chapter 10, we saw how voltages and currents propagate as waves in transmission lines, and we know that the existence of voltages and currents implies the existence of electric and magnetic fields. So we can identify a transmission line as a structure that confines the fields while enabling them to travel along its length as waves. It can be argued that it is the fields that generate the voltage and current in a transmission line wave, and—if there is no structure on which the voltage and current can exist—the fields will exist nevertheless, and will propagate. In free space, the fields are not bounded by any confining structure, and so they may assume *any* magnitude and direction, as initially determined by the device (such as an antenna) that generates them.

When considering electromagnetic waves in free space, we note that the medium is *sourceless* ($\rho_v = \mathbf{J} = 0$). Under these conditions, Maxwell's equations may be written in terms of **E** and **H** only as

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{2}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{3}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{4}$$

Now let us see whether wave motion can be inferred from these four equations without actually solving them. Equation (1) states that if electric field **E** is changing with time at some point, then magnetic field **H** has curl at that point; therefore **H** varies spatially in a direction normal to its orientation direction. Also, if **E** is changing with time, then **H** will in general also change with time, although not necessarily in the same way. Next, we see from Eq. (2) that a time-varying **H** generates **E**, which, having curl, varies spatially in the direction normal to its orientation. We now have once more a changing electric field, our original hypothesis, but this field is present a small distance away from the point of the original disturbance. We might guess (correctly) that the velocity with which the effect moves away from the original point is the velocity of light, but this must be checked by a more detailed examination of Maxwell's equations.

We postulate the existence of a *uniform plane wave,* in which both fields, **E** and **H**, lie in the *transverse plane*—that is, the plane whose normal is the direction of propagation. Furthermore, and by definition, both fields are of constant magnitude in the transverse plane. For this reason, such a wave is sometimes called a *transverse electromagnetic* (TEM) wave. The required spatial variation of both fields in the direction normal to their orientations will therefore occur only in the direction of travel—or normal to the transverse plane. Assume, for example, that $\mathbf{E} = E_x \mathbf{a}_x$, or that the electric field is *polarized* in the x direction. If we further assume that wave travel is in the z direction, we allow spatial variation of **E** only with z. Using Eq. (2), we note that with these restrictions, the curl of **E** reduces to a single term:

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\mu_0 \frac{\partial H_y}{\partial t}\mathbf{a}_y \tag{5}$$

The direction of the curl of **E** in (5) determines the direction of **H**, which we observe to be along the y direction. Therefore, in a uniform plane wave, the directions of **E** and **H** and the direction of travel are mutually orthogonal. Using the y-directed magnetic field, and the fact that it varies only in z, simplifies Eq. (1) to read

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \epsilon_0 \frac{\partial E_x}{\partial t}\mathbf{a}_x \tag{6}$$

Equations (5) and (6) can be more succinctly written:

$$\frac{\partial E_x}{\partial z} = -\mu_0 \frac{\partial H_y}{\partial t} \tag{7}$$

$$\frac{\partial H_y}{\partial z} = -\epsilon_0 \frac{\partial E_x}{\partial t} \tag{8}$$

These equations compare directly with the telegraphist's equations for the lossless transmission line [Eqs. (20) and (21) in Chapter 10]. Further manipulations of (7) and (8) proceed in the same manner as was done with the telegraphist's equations. Specifically, we differentiate (7) with respect to z, obtaining:

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu_0 \frac{\partial^2 H_y}{\partial t \partial z} \tag{9}$$

Then, (8) is differentiated with respect to t:

$$\frac{\partial^2 H_y}{\partial z \partial t} = -\epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \tag{10}$$

Substituting (10) into (9) results in

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \tag{11}$$

This equation, in direct analogy to Eq. (13) in Chapter 10, we identify as the wave equation for our x-polarized TEM electric field in free space. From Eq. (11), we further identify the propagation velocity:

$$\nu = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/s} = c \tag{12}$$

where c denotes the velocity of light in free space. A similar procedure, involving differentiating (7) with t and (8) with z, yields the wave equation for the magnetic field; it is identical in form to (11):

$$\frac{\partial^2 H_y}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 H_y}{\partial t^2} \tag{13}$$

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As was discussed in Chapter 10, the solution to equations of the form of (11) and (13) will be forward- and backward-propagating waves having the general form [in this case for Eq. (11)]:

$$E_x(z, t) = f_1(t - z/\nu) + f_2(t + z/\nu) \tag{14}$$

where again f_1 amd f_2 can be any function whose argument is of the form $t \pm z/\nu$.

From here, we immediately specialize to sinusoidal functions of a specified frequency and write the solution to (11) in the form of forward- and backward-propagating cosines. Because the waves are sinusoidal, we denote their velocity as the *phase velocity,* v_p. The waves are written as:

$$\begin{aligned} E_x(z,t) &= \mathcal{E}_x(z,t) + \mathcal{E}'_x(z,t) \\ &= |E_{x0}| \cos[\omega(t - z/v_p) + \phi_1] + |E'_{x0}| \cos[\omega(t + z/v_p) + \phi_2] \\ &= \underbrace{|E_{x0}| \cos[\omega t - k_0 z + \phi_1]}_{\text{forward } z \text{ travel}} + \underbrace{|E'_{x0}| \cos[\omega t + k_0 z + \phi_2]}_{\text{backward } z \text{ travel}} \end{aligned} \tag{15}$$

In writing the second line of (15), we have used the fact that the waves are traveling in free space, in which case the phase velocity, $v_p = c$. Additionally, the *wavenumber* in free space in defined as

$$\boxed{k_0 \equiv \frac{\omega}{c} \text{ rad/m}} \tag{16}$$

In a manner consistant with our transmission line studies, we refer to the solutions expressed in (15) as the *real instantaneous* forms of the electric field. They are the mathematical representations of what one would experimentally measure. The terms ωt and $k_0 z$, appearing in (15), have units of angle and are usually expressed in radians. We know that ω is the radian time frequency, measuring phase shift *per unit time;* it has units of *rad/s.* In a similar way, we see that k_0 will be interpreted as a *spatial* frequency, which in the present case measures the phase shift *per unit distance* along the z direction in rad/m. We note that k_0 is the phase constant for lossless propagation of uniform plane waves in free space. The *wavelength* in free space is the distance over which the spatial phase shifts by 2π radians, assuming fixed time, or

$$k_0 z = k_0 \lambda = 2\pi \quad \rightarrow \quad \boxed{\lambda = \frac{2\pi}{k_0} \quad \text{(free space)}} \tag{17}$$

The manner in which the waves propagate is the same as we encountered in transmission lines. Specifically, suppose we consider some point (such as a wave crest) on the forward-propagating cosine function of Eq. (15). For a crest to occur, the argument of the cosine must be an integer multiple of 2π. Considering the mth crest of the wave, the condition becomes

$$k_0 z = 2m\pi$$

So let us now consider the point on the cosine that we have chosen, and see what happens as time is allowed to increase. Our requirement is that the entire cosine argument be the same multiple of 2π for all time, in order to keep track of the chosen point. Our condition becomes

$$\omega t - k_0 z = \omega(t - z/c) = 2m\pi \tag{18}$$

As time increases, the position z must also increase in order to satisfy (18). The wave crest (and the entire wave) moves in the positive z direction at phase velocity c (in free space). Using similar reasoning, the wave in Eq. (15) having cosine argument $(\omega t + k_0 z)$ describes a wave that moves in the negative z direction, since as time increases, z must now decrease to keep the argument constant. For simplicity, we will restrict our attention in this chapter to only the positive z traveling wave.

As was done for transmission line waves, we express the real instantaneous fields of Eq. (15) in terms of their phasor forms. Using the forward-propagating field in (15), we write:

$$\mathcal{E}_x(z,t) = \frac{1}{2}\underbrace{|E_{x0}|e^{j\phi_1}}_{E_{x0}} e^{-jk_0 z} e^{j\omega t} + c.c. = \frac{1}{2} E_{xs} e^{j\omega t} + c.c. = \text{Re}[E_{xs} e^{j\omega t}] \tag{19}$$

where $c.c.$ denotes the complex conjugate, and where we identify the *phasor electric field* as $E_{xs} = E_{x0} e^{-jk_0 z}$. As indicated in (19), E_{x0} is the *complex* amplitude (which includes the phase, ϕ_1).

EXAMPLE 11.1

Let us express $\mathcal{E}_y(z,t) = 100\cos(10^8 t - 0.5z + 30°)$ V/m as a phasor.

Solution. We first go to exponential notation,

$$\mathcal{E}_y(z,t) = \text{Re}\left[100 e^{\,j(10^8 t - 0.5z + 30°)}\right]$$

and then drop Re and suppress $e^{j10^8 t}$, obtaining the phasor

$$E_{ys}(z) = 100 e^{-j0.5z + j30°}$$

Note that a mixed nomenclature is used for the angle in this case; that is, $0.5z$ is in radians, while 30° is in degrees. Given a scalar component or a vector expressed as a phasor, we may easily recover the time-domain expression.

EXAMPLE 11.2

Given the complex amplitude of the electric field of a uniform plane wave, $\mathbf{E}_0 = 100\mathbf{a}_x + 20\angle 30°\mathbf{a}_y$ V/m, construct the phasor and real instantaneous fields if the wave is known to propagate in the forward z direction in free space and has frequency of 10 MHz.

Solution. We begin by constructing the general phasor expression:

$$\mathbf{E}_s(z) = \left[100\mathbf{a}_x + 20 e^{j30°}\mathbf{a}_y\right] e^{-jk_0 z}$$

where $k_0 = \omega/c = 2\pi \times 10^7 / 3 \times 10^8 = 0.21$ rad/m. The real instantaneous form is then found through the rule expressed in Eq. (19):

$$\begin{aligned}\mathcal{E}(z,t) &= \text{Re}\left[100 e^{-j0.21z} e^{j2\pi\times 10^7 t}\mathbf{a}_x + 20 e^{j30°} e^{-j0.21z} e^{j2\pi\times 10^7 t}\mathbf{a}_y\right] \\ &= \text{Re}\left[100 e^{j(2\pi\times 10^7 t - 0.21z)}\mathbf{a}_x + 20 e^{j(2\pi\times 10^7 t - 0.21z + 30°)}\mathbf{a}_y\right] \\ &= 100\cos(2\pi\times 10^7 t - 0.21z)\mathbf{a}_x + 20\cos(2\pi\times 10^7 t - 0.21z + 30°)\,\mathbf{a}_y\end{aligned}$$

It is evident that taking the partial derivative of any field quantity with respect to time is equivalent to multiplying the corresponding phasor by $j\omega$. As an example, we can express Eq. (8) (using sinusoidal fields) as

$$\frac{\partial \mathcal{H}_y}{\partial z} = -\epsilon_0 \frac{\partial \mathcal{E}_x}{\partial t} \tag{20}$$

where, in a manner consistent with (19):

$$\mathcal{E}_x(z, t) = \frac{1}{2} E_{xs}(z)\, e^{j\omega t} + c.c. \qquad \text{and} \qquad \mathcal{H}_y(z, t) = \frac{1}{2} H_{ys}(z)\, e^{j\omega t} + c.c. \tag{21}$$

On substituting the fields in (21) into (20), the latter equation simplifies to

$$\boxed{\frac{dH_{ys}(z)}{dz} = -j\omega\epsilon_0 E_{xs}(z)} \tag{22}$$

In obtaining this equation, we note first that the complex conjugate terms in (21) produce their own separate equation, redundant with (22); second, the $e^{j\omega t}$ factors, common to both sides, have divided out; third, the partial derivative with z becomes the total derivative, since the phasor, H_{ys}, depends only on z.

We next apply this result to Maxwell's equations, to obtain them in phasor form. Substituting the field as expressed in (21) into Eqs. (1) through (4) results in

$$\nabla \times \mathbf{H}_s = j\omega\epsilon_0 \mathbf{E}_s \tag{23}$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu_0 \mathbf{H}_s \tag{24}$$

$$\nabla \cdot \mathbf{E}_s = 0 \tag{25}$$

$$\nabla \cdot \mathbf{H}_s = 0 \tag{26}$$

It should be noted that (25) and (26) are no longer independent relationships, for they can be obtained by taking the divergence of (23) and (24), respectively.

Eqs. (23) through (26) may be used to obtain the sinusoidal steady-state vector form of the wave equation in free space. We begin by taking the curl of both sides of (24):

$$\nabla \times \nabla \times \mathbf{E}_s = -j\omega\mu_0 \nabla \times \mathbf{H}_s = \nabla(\nabla \cdot \mathbf{E}_s) - \nabla^2 \mathbf{E}_s \tag{27}$$

where the last equality is an identity, which defines the *vector Laplacian* of $\mathbf{E}_s$:

$$\boxed{\nabla^2 \mathbf{E}_s = \nabla(\nabla \cdot \mathbf{E}_s) - \nabla \times \nabla \times \mathbf{E}_s}$$

From (25), we note that $\nabla \cdot \mathbf{E}_s = 0$. Using this, and substituting (23) in (27), we obtain

$$\boxed{\nabla^2 \mathbf{E}_s = -k_0^2 \mathbf{E}_s} \tag{28}$$

where again, $k_0 = \omega/c = \omega\sqrt{\mu_0\epsilon_0}$. Equation (28) is known as the vector Helmholtz equation in free space.[1] It is fairly formidable when expanded, even in rectangular coordinates, for three scalar phasor equations result (one for each vector component), and each equation has four terms. The x component of (28) becomes, still using the del-operator notation,

$$\boxed{\nabla^2 E_{xs} = -k_0^2 E_{xs}} \tag{29}$$

and the expansion of the operator leads to the second-order partial differential equation

$$\frac{\partial^2 E_{xs}}{\partial x^2} + \frac{\partial^2 E_{xs}}{\partial y^2} + \frac{\partial^2 E_{xs}}{\partial z^2} = -k_0^2 E_{xs}$$

Again, assuming a uniform plane wave in which E_{xs} does not vary with x or y, the two corresponding derivatives are zero, and we obtain

$$\boxed{\frac{d^2 E_{xs}}{dz^2} = -k_0^2 E_{xs}} \tag{30}$$

the solution of which we already know:

$$\boxed{E_{xs}(z) = E_{x0}e^{-jk_0z} + E'_{x0}e^{jk_0z}} \tag{31}$$

Let us now return to Maxwell's equations, (23) through (26), and determine the form of the **H** field. Given $\mathbf{E}_s$, $\mathbf{H}_s$ is most easily obtained from (24):

$$\nabla \times \mathbf{E}_s = -j\omega\mu_0\mathbf{H}_s \tag{24}$$

which is greatly simplified for a single E_{xs} component varying only with z,

$$\frac{dE_{xs}}{dz} = -j\omega\mu_0 H_{ys}$$

Using (31) for E_{xs}, we have

$$H_{ys} = -\frac{1}{j\omega\mu_0}\left[(-jk_0)E_{x0}e^{-jk_0z} + (jk_0)E'_{x0}e^{jk_0z}\right]$$

$$= E_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}e^{-jk_0z} - E'_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}e^{jk_0z} = H_{y0}e^{-jk_0z} + H'_{y0}e^{jk_0z} \tag{32}$$

In real instantaneous form, this becomes

$$\boxed{H_y(z,t) = E_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}\cos(\omega t - k_0z) - E'_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}\cos(\omega t + k_0z)} \tag{33}$$

where E_{x0} and E'_{x0} are assumed real.

[1] Hermann Ludwig Ferdinand von Helmholtz (1821–1894) was a professor at the University of Berlin working in the fields of physiology, electrodynamics, and optics. Hertz was one of his students.

In general, we find from (32) that the electric and magnetic field amplitudes of the forward-propagating wave in free space are related through

$$E_{x0} = \sqrt{\frac{\mu_0}{\epsilon_0}} H_{y0} = \eta_0 H_{y0} \tag{34a}$$

We also find the backward-propagating wave amplitudes are related through

$$E'_{x0} = -\sqrt{\frac{\mu_0}{\epsilon_0}} H'_{y0} = -\eta_0 H'_{y0} \tag{34b}$$

where the *intrinsic impedance* of free space is defined as

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \doteq 120\pi \ \Omega \tag{35}$$

The dimension of η_0 in ohms is immediately evident from its definition as the ratio of E (in units of V/m) to H (in units of A/m). It is in direct analogy to the characteristic impedance, Z_0, of a transmission line, where we defined the latter as the ratio of voltage to current in a traveling wave. We note that the difference between (34*a*) and (34*b*) is a minus sign. This is consistent with the transmission line analogy that led to Eqs. (25*a*) and (25*b*) in Chapter 10. Those equations accounted for the definitions of positive and negative current associated with forward and backward voltage waves. In a similar way, Eq. (34*a*) specifies that in a forward-z propagating uniform plane wave whose electric field vector lies in the positive x direction at a given point in time and space, the magnetic field vector lies in the *positive* y direction at the same space and time coordinates. In the case of a backward-z propagating wave having a positive x-directed electric field, the magnetic field vector lies in the *negative* y direction. The physical significance of this has to do with the definition of power flow in the wave, as specified through the Poynting vector, $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ (in watts/m^2). The cross product of $\mathbf{E}$ with $\mathbf{H}$ must give the correct wave propagation direction, and so the need for the minus sign in (34*b*) is apparent. Issues relating to power transmission will be addressed in Section 11.3.

Some feeling for the way in which the fields vary in space may be obtained from Figures 11.1*a* and 11.1*b*. The electric field intensity in Figure 11.1*a* is shown at $t = 0$, and the instantaneous value of the field is depicted along three lines, the z axis and arbitrary lines parallel to the z axis in the $x = 0$ and $y = 0$ planes. Since the field is uniform in planes perpendicular to the z axis, the variation along all three of the lines is the same. One complete cycle of the variation occurs in a wavelength, λ. The values of H_y at the same time and positions are shown in Figure 11.1*b*.

A uniform plane wave cannot exist physically, for it extends to infinity in two dimensions at least and represents an infinite amount of energy. The distant field of a transmitting antenna, however, is essentially a uniform plane wave in some limited region; for example, a radar signal impinging on a distant target is closely a uniform plane wave.

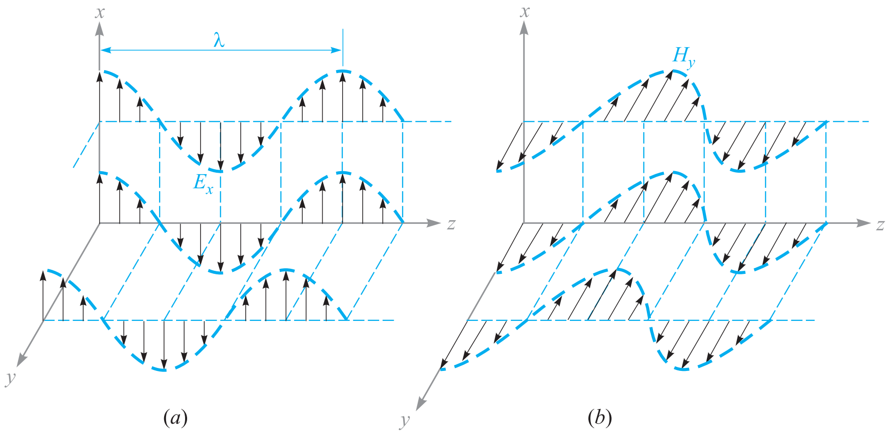

Figure 11.1 (*a*) Arrows represent the instantaneous values of $E_{x0}\cos[\omega(t - z/c)]$ at $t = 0$ along the z axis, along an arbitrary line in the $x = 0$ plane parallel to the z axis, and along an arbitrary line in the $y = 0$ plane parallel to the z axis. (*b*) Corresponding values of H_y are indicated. Note that E_x and H_y are in phase at any point in time.

Although we have considered only a wave varying sinusoidally in time and space, a suitable combination of solutions to the wave equation may be made to achieve a wave of any desired form, but which satisfies (14). The summation of an infinite number of harmonics through the use of a Fourier series can produce a periodic wave of square or triangular shape in both space and time. Nonperiodic waves may be obtained from our basic solution by Fourier integral methods. These topics are among those considered in the more advanced books on electromagnetic theory.

WW Animations

D11.1. The electric field amplitude of a uniform plane wave propagating in the $\mathbf{a}_z$ direction is 250 V/m. If $\mathbf{E} = E_x\mathbf{a}_x$ and $\omega = 1.00$ Mrad/s, find: (*a*) the frequency; (*b*) the wavelength; (*c*) the period; (*d*) the amplitude of **H**.

Ans. 159 kHz; 1.88 km; 6.28 μs; 0.663 A/m

D11.2. Let $\mathbf{H}_s = (2\angle{-40^\circ}\mathbf{a}_x - 3\angle 20^\circ\mathbf{a}_y)e^{-j0.07z}$ A/m for a uniform plane wave traveling in free space. Find: (*a*) ω; (*b*) H_x at $P(1, 2, 3)$ at $t = 31$ ns; (*c*) $|\mathbf{H}|$ at $t = 0$ at the origin.

Ans. 21.0 Mrad/s; 1.934 A/m; 3.22 A/m

11.2 WAVE PROPAGATION IN DIELECTRICS

We now extend our analytical treatment of the uniform plane wave to propagation in a dielectric of permittivity ϵ and permeability μ. The medium is assumed to be homogeneous (having constant μ and ϵ with position) and isotropic (in which μ and

ϵ are invariant with field orientation). The Helmholtz equation is

$$\nabla^2 \mathbf{E}_s = -k^2 \mathbf{E}_s \tag{36}$$

where the wavenumber is a function of the material properties, as described by μ and ϵ:

$$k = \omega\sqrt{\mu\epsilon} = k_0\sqrt{\mu_r \epsilon_r} \tag{37}$$

For E_{xs} we have

$$\frac{d^2 E_{xs}}{dz^2} = -k^2 E_{xs} \tag{38}$$

An important feature of wave propagation in a dielectric is that k can be complex-valued, and as such it is referred to as the complex *propagation constant*. A general solution of (38), in fact, allows the possibility of a complex k, and it is customary to write it in terms of its real and imaginary parts in the following way:

$$jk = \alpha + j\beta \tag{39}$$

A solution to (38) will be:

$$E_{xs} = E_{x0}e^{-jkz} = E_{x0}e^{-\alpha z}e^{-j\beta z} \tag{40}$$

Multiplying (40) by $e^{j\omega t}$ and taking the real part yields a form of the field that can be more easily visualized:

$$E_x = E_{x0}e^{-\alpha z}\cos(\omega t - \beta z) \tag{41}$$

We recognize this as a uniform plane wave that propagates in the forward z direction with phase constant β, but which (for positive α) loses amplitude with increasing z according to the factor $e^{-\alpha z}$. Thus the general effect of a complex-valued k is to yield a traveling wave that changes its amplitude with distance. If α is positive, it is called the *attenuation coefficient*. If α is negative, the wave grows in amplitude with distance, and α is called the *gain coefficient*. The latter effect would occur, for example, in laser amplifiers. In the present and future discussions in this book, we will consider only passive media, in which one or more loss mechanisms are present, thus producing a positive α.

The attenuation coefficient is measured in nepers per meter (Np/m) so that the exponent of e can be measured in the dimensionless units of nepers. Thus, if $\alpha = 0.01$ Np/m, the crest amplitude of the wave at $z = 50$ m will be $e^{-0.5}/e^{-0} = 0.607$ of its value at $z = 0$. In traveling a distance $1/\alpha$ in the $+z$ direction, the amplitude of the wave is reduced by the familiar factor of e^{-1}, or 0.368.

The ways in which physical processes in a material can affect the wave electric field are described through a *complex permittivity* of the form

$$\epsilon = \epsilon' - j\epsilon'' = \epsilon_0(\epsilon'_r - j\epsilon''_r) \tag{42}$$

Two important mechanisms that give rise to a complex permittivity (and thus result in wave losses) are bound electron or ion oscillations and dipole relaxation, both of which are discussed in Appendix E. An additional mechanism is the conduction of free electrons or holes, which we will explore at length in this chapter.

Losses arising from the response of the medium to the magnetic field can occur as well, and these are modeled through a *complex permeability,* $\mu = \mu' - j\mu'' = \mu_0(\mu'_r - j\mu''_r)$. Examples of such media include *ferrimagnetic* materials, or *ferrites*. The magnetic response is usually very weak compared to the dielectric response in most materials of interest for wave propagation; in such materials $\mu \approx \mu_0$. Consequently, our discussion of loss mechanisms will be confined to those described through the complex permittivity, and we will assume that μ is entirely real in our treatment.

We can substitute (42) into (37), which results in

$$k = \omega\sqrt{\mu(\epsilon' - j\epsilon'')} = \omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\epsilon''}{\epsilon'}} \tag{43}$$

Note the presence of the second radical factor in (43), which becomes unity (and real) as ϵ'' vanishes. With nonzero ϵ'', k is complex, and so losses occur which are quantified through the attenuation coefficient, α, in (39). The phase constant, β (and consequently the wavelength and phase velocity), will also be affected by ϵ''. α and β are found by taking the real and imaginary parts of jk from (43). We obtain:

$$\alpha = \mathrm{Re}\{jk\} = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left(\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right)^{1/2} \tag{44}$$

$$\beta = \mathrm{Im}\{jk\} = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left(\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right)^{1/2} \tag{45}$$

We see that a nonzero α (and hence loss) results if the imaginary part of the permittivity, ϵ'', is present. We also observe in (44) and (45) the presence of the ratio ϵ''/ϵ', which is called the *loss tangent*. The meaning of the term will be demonstrated when we investigate the specific case of conductive media. The practical importance of the ratio lies in its magnitude compared to unity, which enables simplifications to be made in (44) and (45).

Whether or not losses occur, we see from (41) that the wave phase velocity is given by

$$v_p = \frac{\omega}{\beta} \tag{46}$$

The wavelength is the distance required to effect a phase change of 2π radians

$$\beta\lambda = 2\pi$$

which leads to the fundamental definition of wavelength,

$$\lambda = \frac{2\pi}{\beta} \tag{47}$$

Because we have a uniform plane wave, the magnetic field is found through

$$H_{ys} = \frac{E_{x0}}{\eta} e^{-\alpha z} e^{-j\beta z}$$

where the intrinsic impedance is now a complex quantity,

$$\eta = \sqrt{\frac{\mu}{\epsilon' - j\epsilon''}} = \sqrt{\frac{\mu}{\epsilon'}} \frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}} \tag{48}$$

The electric and magnetic fields are no longer in phase.

A special case is that of a lossless medium, or *perfect dielectric,* in which $\epsilon'' = 0$, and so $\epsilon = \epsilon'$. From (44), this leads to $\alpha = 0$, and from (45),

$$\beta = \omega\sqrt{\mu\epsilon'} \qquad \text{(lossless medium)} \tag{49}$$

With $\alpha = 0$, the real field assumes the form

$$E_x = E_{x0} \cos(\omega t - \beta z) \tag{50}$$

We may interpret this as a wave traveling in the $+z$ direction at a phase velocity v_p, where

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon'}} = \frac{c}{\sqrt{\mu_r \epsilon'_r}}$$

The wavelength is

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\epsilon'}} = \frac{1}{f\sqrt{\mu\epsilon'}} = \frac{c}{f\sqrt{\mu_r\epsilon'_r}} = \frac{\lambda_0}{\sqrt{\mu_r\epsilon'_r}} \qquad \text{(lossless medium)} \tag{51}$$

where λ_0 is the free space wavelength. Note that $\mu_r \epsilon'_r > 1$, and therefore the wavelength is shorter and the velocity is lower in all real media than they are in free space.

Associated with E_x is the magnetic field intensity

$$H_y = \frac{E_{x0}}{\eta} \cos(\omega t - \beta z)$$

where the intrinsic impedance is

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \tag{52}$$

The two fields are once again perpendicular to each other, perpendicular to the direction of propagation, and in phase with each other everywhere. Note that when **E** is crossed into **H**, the resultant vector is in the direction of propagation. We shall see the reason for this when we discuss the Poynting vector.

EXAMPLE 11.3

Let us apply these results to a 1-MHz plane wave propagating in fresh water. At this frequency, losses in water are negligible, which means that we can assume that $\epsilon'' \doteq 0$. In water, $\mu_r = 1$ and at 1 MHz, $\epsilon'_r = 81$.

Solution. We begin by calculating the phase constant. Using (45) with $\epsilon'' = 0$, we have

$$\beta = \omega\sqrt{\mu\epsilon'} = \omega\sqrt{\mu_0\epsilon_0}\sqrt{\epsilon'_r} = \frac{\omega\sqrt{\epsilon'_r}}{c} = \frac{2\pi \times 10^6\sqrt{81}}{3.0 \times 10^8} = 0.19 \text{ rad/m}$$

Using this result, we can determine the wavelength and phase velocity:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{.19} = 33 \text{ m}$$

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 10^6}{.19} = 3.3 \times 10^7 \text{ m/s}$$

The wavelength in air would have been 300 m. Continuing our calculations, we find the intrinsic impedance using (48) with $\epsilon'' = 0$:

$$\eta = \sqrt{\frac{\mu}{\epsilon'}} = \frac{\eta_0}{\sqrt{\epsilon'_r}} = \frac{377}{9} = 42\ \Omega$$

If we let the electric field intensity have a maximum amplitude of 0.1 V/m, then

$$E_x = 0.1\cos(2\pi 10^6 t - .19z) \text{ V/m}$$

$$H_y = \frac{E_x}{\eta} = (2.4 \times 10^{-3})\cos(2\pi 10^6 t - .19z) \text{ A/m}$$

D11.3. A 9.375-GHz uniform plane wave is propagating in polyethylene (see Appendix C). If the amplitude of the electric field intensity is 500 V/m and the material is assumed to be lossless, find: (*a*) the phase constant; (*b*) the wavelength in the polyethylene; (*c*) the velocity of propagation; (*d*) the intrinsic impedance; (*e*) the amplitude of the magnetic field intensity.

Ans. 295 rad/m; 2.13 cm; 1.99×10^8 m/s; 251 Ω; 1.99 A/m

EXAMPLE 11.4

We again consider plane wave propagation in water, but at the much higher microwave frequency of 2.5 GHz. At frequencies in this range and higher, dipole relaxation and resonance phenomena in the water molecules become important.[2] Real and imaginary parts of the permittivity are present, and both vary with frequency. At frequencies below that of visible light, the two mechanisms together produce a value of ϵ'' that increases with increasing frequency, reaching a maximum in the vicinity of 10^{13} Hz. ϵ' decreases with increasing frequency, reaching a minimum also in the vicinity of 10^{13} Hz. Reference 3 provides specific details. At 2.5 GHz, dipole relaxation effects dominate. The permittivity values are $\epsilon'_r = 78$ and $\epsilon''_r = 7$. From (44), we have

$$\alpha = \frac{(2\pi \times 2.5 \times 10^9)\sqrt{78}}{(3.0 \times 10^8)\sqrt{2}} \left(\sqrt{1 + \left(\frac{7}{78}\right)^2} - 1 \right)^{1/2} = 21 \text{ Np/m}$$

This first calculation demonstrates the operating principle of the *microwave oven*. Almost all foods contain water, and so they can be cooked when incident microwave radiation is absorbed and converted into heat. Note that the field will attenuate to a value of e^{-1} times its initial value at a distance of $1/\alpha = 4.8$ cm. This distance is called the *penetration depth* of the material, and of course it is frequency-dependent. The 4.8 cm depth is reasonable for cooking food, since it would lead to a temperature rise that is fairly uniform throughout the depth of the material. At much higher frequencies, where ϵ'' is larger, the penetration depth decreases, and too much power is absorbed at the surface; at lower frequencies, the penetration depth increases, and not enough overall absorption occurs. Commercial microwave ovens operate at frequencies in the vicinity of 2.5 GHz.

Using (45), in a calculation very similar to that for α, we find $\beta = 464$ rad/m. The wavelength is $\lambda = 2\pi/\beta = 1.4$ cm, whereas in free space this would have been $\lambda_0 = c/f = 12$ cm.

Using (48), the intrinsic impedance is found to be

$$\eta = \frac{377}{\sqrt{78}} \frac{1}{\sqrt{1 - j(7/78)}} = 43 + j1.9 = 43\angle 2.6^\circ \ \Omega$$

and E_x leads H_y in time by 2.6° at every point.

We next consider the case of conductive materials, in which currents are formed by the motion of free electrons or holes under the influence of an electric field. The governing relation is $\mathbf{J} = \sigma \mathbf{E}$, where σ is the material conductivity. With finite conductivity, the wave loses power through resistive heating of the material. We look for an interpretation of the complex permittivity as it relates to the conductivity.

[2] These mechanisms and how they produce a complex permittivity are described in Appendix D. Additionally, the reader is referred to pp. 73–84 in Reference 1 and pp. 678–82 in Reference 2 for general treatments of relaxation and resonance effects on wave propagation. Discussions and data that are specific to water are presented in Reference 3, pp. 314–16.

Consider the Maxwell curl equation (23) which, using (42), becomes:

$$\nabla \times \mathbf{H}_s = j\omega(\epsilon' - j\epsilon'')\mathbf{E}_s = \omega\epsilon''\mathbf{E}_s + j\omega\epsilon'\mathbf{E}_s \tag{53}$$

This equation can be expressed in a more familiar way, in which conduction current is included:

$$\nabla \times \mathbf{H}_s = \mathbf{J}_s + j\omega\epsilon\mathbf{E}_s \tag{54}$$

We next use $\mathbf{J}_s = \sigma\mathbf{E}_s$, and interpret ϵ in (54) as ϵ'. The latter equation becomes:

$$\boxed{\nabla \times \mathbf{H}_s = (\sigma + j\omega\epsilon')\mathbf{E}_s = \mathbf{J}_{\sigma s} + \mathbf{J}_{ds}} \tag{55}$$

which we have expressed in terms of conduction current density, $\mathbf{J}_{\sigma s} = \sigma\mathbf{E}_s$, and displacement current density, $\mathbf{J}_{ds} = j\omega\epsilon'\mathbf{E}_s$. Comparing Eqs. (53) and (55), we find that in a conductive medium:

$$\boxed{\epsilon'' = \frac{\sigma}{\omega}} \tag{56}$$

Let us now turn our attention to the case of a dielectric material in which the loss is very small. The criterion by which we should judge whether or not the loss is small is the magnitude of the loss tangent, ϵ''/ϵ'. This parameter will have a direct influence on the attenuation coefficient, α, as seen from Eq. (44). In the case of conducting media, to which (56) applies, the loss tangent becomes $\sigma/\omega\epsilon'$. By inspecting (55), we see that the ratio of conduction current density to displacement current density magnitudes is

$$\boxed{\frac{J_{\sigma s}}{J_{ds}} = \frac{\epsilon''}{j\epsilon'} = \frac{\sigma}{j\omega\epsilon'}} \tag{57}$$

That is, these two vectors point in the same direction in space, but they are 90° out of phase in time. Displacement current density leads conduction current density by 90°, just as the current through a capacitor leads the current through a resistor in parallel with it by 90° in an ordinary electric circuit. This phase relationship is shown in Figure 11.2. The angle θ (not to be confused with the polar angle in spherical coordinates) may therefore be identified as the angle by which the displacement current density leads the total current density, and

$$\boxed{\tan\theta = \frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon'}} \tag{58}$$

The reasoning behind the term *loss tangent* is thus evident. Problem 11.16 at the end of the chapter indicates that the Q of a capacitor (its quality factor, not its charge) that incorporates a lossy dielectric is the reciprocal of the loss tangent.

If the loss tangent is small, then we may obtain useful approximations for the attenuation and phase constants, and the intrinsic impedance. The criterion for a small loss tangent is $\epsilon''/\epsilon' \ll 1$, which we say identifies the medium as a *good dielectric*.

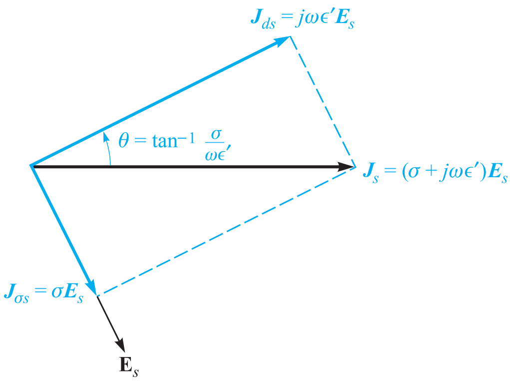

Figure 11.2 The time-phase relationship between $\mathbf{J}_{ds}$, $\mathbf{J}_{\sigma s}$, $\mathbf{J}_s$, and $\mathbf{E}_s$. The tangent of θ is equal to $\sigma/\omega\epsilon'$, and $90° - \theta$ is the common power-factor angle, or the angle by which $\mathbf{J}_s$ leads $\mathbf{E}_s$.

Considering a conductive material, for which $\epsilon'' = \sigma/\omega$, (43) becomes

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\sigma}{\omega\epsilon'}} \tag{59}$$

We may expand the second radical using the binomial theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

where $|x| \ll 1$. We identify x as $-j\sigma/\omega\epsilon'$ and n as $1/2$, and thus

$$jk = j\omega\sqrt{\mu\epsilon'}\left[1 - j\frac{\sigma}{2\omega\epsilon'} + \frac{1}{8}\left(\frac{\sigma}{\omega\epsilon'}\right)^2 + \cdots\right] = \alpha + j\beta$$

Now, for a good dielectric,

$$\alpha = \text{Re}(jk) \doteq j\omega\sqrt{\mu\epsilon'}\left(-j\frac{\sigma}{2\omega\epsilon'}\right) = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon'}} \tag{60a}$$

and

$$\beta = \text{Im}(jk) \doteq \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\epsilon'}\right)^2\right] \tag{60b}$$

Equations (60*a*) and (60*b*) can be compared directly with the transmission line α and β under low-loss conditions, as expressed in Eqs. (54*a*) and (55*b*) in Chapter 10. In this comparison, we associate σ with G, μ with L, and ϵ with C. Note that in plane wave propagation in media with no boundaries, there can be no quantity that is analogous to the transmission line conductor resistance parameter, R. In many cases, the second term in (60*b*) is small enough, so that

$$\beta \doteq \omega\sqrt{\mu\epsilon'} \tag{61}$$

Applying the binomial expansion to (48), we obtain, for a good dielectric

$$\eta \doteq \sqrt{\frac{\mu}{\epsilon'}}\left[1 - \frac{3}{8}\left(\frac{\sigma}{\omega\epsilon'}\right)^2 + j\frac{\sigma}{2\omega\epsilon'}\right] \tag{62a}$$

or

$$\eta \doteq \sqrt{\frac{\mu}{\epsilon'}}\left(1 + j\frac{\sigma}{2\omega\epsilon'}\right) \tag{62b}$$

The conditions under which these approximations can be used depend on the desired accuracy, measured by how much the results deviate from those given by the exact formulas, (44) and (45). Deviations of no more than a few percent occur if $\sigma/\omega\epsilon' < 0.1$.

EXAMPLE 11.5

As a comparison, we repeat the computations of Example 11.4, using the approximation formulas (60*a*), (61), and (62*b*).

Solution. First, the loss tangent in this case is $\epsilon''/\epsilon' = 7/78 = 0.09$. Using (60), with $\epsilon'' = \sigma/\omega$, we have

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{1}{2}(7 \times 8.85 \times 10^{12})(2\pi \times 2.5 \times 10^9)\frac{377}{\sqrt{78}} = 21 \text{ cm}^{-1}$$

We then have, using (61*b*),

$$\beta \doteq (2\pi \times 2.5 \times 10^9)\sqrt{78}/(3 \times 10^8) = 464 \text{ rad/m}$$

Finally, with (62*b*),

$$\eta \doteq \frac{377}{\sqrt{78}}\left(1 + j\frac{7}{2 \times 78}\right) = 43 + j1.9$$

These results are identical (within the accuracy limitations as determined by the given numbers) to those of Example 11.4. Small deviations will be found, as the reader can verify by repeating the calculations of both examples and expressing the results to four or five significant figures. As we know, this latter practice would not be meaningful because the given parameters were not specified with such accuracy. Such is often the case, since measured values are not always known with high precision. Depending on how precise these values are, one can sometimes use a more relaxed judgment on when the approximation formulas can be used by allowing loss tangent values that can be larger than 0.1 (but still less than 1).

D11.4. Given a nonmagnetic material having $\epsilon'_r = 3.2$ and $\sigma = 1.5 \times 10^{-4}$ S/m, find numerical values at 3 MHz for the (*a*) loss tangent; (*b*) attenuation constant; (*c*) phase constant; (*d*) intrinsic impedance.

Ans. 0.28; 0.016 Np/m; 0.11 rad/m; $207\angle 7.8^\circ\ \Omega$

D11.5. Consider a material for which $\mu_r = 1$, $\epsilon'_r = 2.5$, and the loss tangent is 0.12. If these three values are constant with frequency in the range $0.5\text{ MHz} \le f \le 100$ MHz, calculate: (*a*) σ at 1 and 75 MHz; (*b*) λ at 1 and 75 MHz; (*c*) v_p at 1 and 75 MHz.

Ans. 1.67×10^{-5} and 1.25×10^{-3} S/m; 190 and 2.53 m; 1.90×10^8 m/s twice

11.3 POYNTING'S THEOREM AND WAVE POWER

WW Illustrations

In order to find the power flow associated with an electromagnetic wave, it is necessary to develop a power theorem for the electromagnetic field known as the Poynting theorem. It was originally postulated in 1884 by an English physicist, John H. Poynting.

The development begins with one of Maxwell's curl equations, in which we assume that the medium may be conductive:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{63}$$

Next, we take the scalar product of both sides of (63) with **E**,

$$\mathbf{E} \cdot \nabla \times \mathbf{H} = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{64}$$

We then introduce the following vector identity, which may be proved by expansion in rectangular coordinates:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{E} \cdot \nabla \times \mathbf{H} + \mathbf{H} \cdot \nabla \times \mathbf{E} \tag{65}$$

Using (65) in the left side of (64) results in

$$\mathbf{H} \cdot \nabla \times \mathbf{E} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{66}$$

where the curl of the electric field is given by the other Maxwell curl equation:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Therefore

$$-\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}$$

or

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} \tag{67}$$

The two time derivatives in (67) can be rearranged as follows:

$$\epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{D} \cdot \mathbf{E}\right) \tag{68a}$$

and

$$\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B} \cdot \mathbf{H}\right) \tag{68b}$$

With these, Eq. (67) becomes

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{J} \cdot \mathbf{E} + \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{D} \cdot \mathbf{E}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B} \cdot \mathbf{H}\right) \tag{69}$$

Finally, we integrate (69) throughout a volume:

$$-\int_{\text{vol}} \nabla \cdot (\mathbf{E} \times \mathbf{H})\, dv = \int_{\text{vol}} \mathbf{J} \cdot \mathbf{E}\, dv + \int_{\text{vol}} \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{D} \cdot \mathbf{E}\right) dv + \int_{\text{vol}} \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B} \cdot \mathbf{H}\right) dv$$

The divergence theorem is then applied to the left-hand side, thus converting the volume integral there into an integral over the surface that encloses the volume. On the right-hand side, the operations of spatial integration and time differentiation are interchanged. The final result is

$$-\oint_{\text{area}} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = \int_{\text{vol}} \mathbf{J} \cdot \mathbf{E}\, dv + \frac{d}{dt}\int_{\text{vol}} \frac{1}{2}\mathbf{D} \cdot \mathbf{E}\, dv + \frac{d}{dt}\int_{\text{vol}} \frac{1}{2}\mathbf{B} \cdot \mathbf{H}\, dv \tag{70}$$

Equation (70) is known as Poynting's theorem. On the right-hand side, the first integral is the total (but instantaneous) ohmic power dissipated within the volume. The second integral is the total energy stored in the electric field, and the third integral is the stored energy in the magnetic field.[3] Since time derivatives are taken of the second and third integrals, those results give the time rates of increase of energy stored within the volume, or the instantaneous power going to increase the stored energy. The sum of the expressions on the right must therefore be the total power flowing *into* this volume, and so the total power flowing *out* of the volume is

$$\oint_{\text{area}} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} \quad \text{W} \tag{71}$$

where the integral is over the closed surface surrounding the volume. The cross product $\mathbf{E} \times \mathbf{H}$ is known as the Poynting vector, $\mathbf{S}$,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \quad \text{W/m}^2 \tag{72}$$

which is interpreted as an instantaneous power density, measured in watts per square meter (W/m^2). The direction of the vector $\mathbf{S}$ indicates the direction of the instantaneous

[3] This is the expression for magnetic field energy that we have been anticipating since Chapter 8.

power flow at a point, and many of us think of the Poynting vector as a "pointing" vector. This homonym, while accidental, is correct.[4]

Because **S** is given by the cross product of **E** and **H**, the direction of power flow at any point is normal to both the **E** and **H** vectors. This certainly agrees with our experience with the uniform plane wave, for propagation in the $+z$ direction was associated with an E_x and H_y component,

$$E_x \mathbf{a}_x \times H_y \mathbf{a}_y = S_z \mathbf{a}_z$$

In a perfect dielectric, the **E** and **H** field amplitudes are given by

$$E_x = E_{x0} \cos(\omega t - \beta z)$$

$$H_y = \frac{E_{x0}}{\eta} \cos(\omega t - \beta z)$$

where η is real. The power density amplitude is therefore

$$S_z = \frac{E_{x0}^2}{\eta} \cos^2(\omega t - \beta z) \tag{73}$$

In the case of a lossy dielectric, E_x and H_y are not in time phase. We have

$$E_x = E_{x0} e^{-\alpha z} \cos(\omega t - \beta z)$$

If we let

$$\eta = |\eta| \angle \theta_\eta$$

then we may write the magnetic field intensity as

$$H_y = \frac{E_{x0}}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)$$

Thus,

$$S_z = E_x H_y = \frac{E_{x0}^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta) \tag{74}$$

Because we are dealing with a sinusoidal signal, the time-average power density, $\langle S_z \rangle$, is the quantity that will ultimately be measured. To find this, we integrate (74) over one cycle and divide by the period $T = 1/f$. Additionally, the identity $\cos A \cos B = 1/2 \cos(A + B) + 1/2 \cos(A - B)$ is applied to the integrand, and we obtain:

$$\langle S_z \rangle = \frac{1}{T} \int_0^T \frac{1}{2} \frac{E_{x0}^2}{|\eta|} e^{-2\alpha z} [\cos(2\omega t - 2\beta z - 2\theta_\eta) + \cos \theta_\eta] \, dt \tag{75}$$

[4] Note that the vector symbol **S** is used for the Poynting vector, and is not to be confused with the differential area vector, $d\mathbf{S}$. The latter, as we know, is the product of the outward normal to the surface and the differential area.

The second-harmonic component of the integrand in (75) integrates to zero, leaving only the contribution from the dc component. The result is

$$\langle S_z \rangle = \frac{1}{2}\frac{E_{x0}^2}{|\eta|}e^{-2\alpha z}\cos\theta_\eta \tag{76}$$

Note that the power density attenuates as $e^{-2\alpha z}$, whereas E_x and H_y fall off as $e^{-\alpha z}$.

We may finally observe that the preceding expression can be obtained very easily by using the phasor forms of the electric and magnetic fields. In vector form, this is

$$\langle \mathbf{S} \rangle = \frac{1}{2}\text{Re}(\mathbf{E}_s \times \mathbf{H}_s^*) \quad \text{W/m}^2 \tag{77}$$

In the present case

$$\mathbf{E}_s = E_{x0}e^{-j\beta z}\mathbf{a}_x$$

and

$$\mathbf{H}_s^* = \frac{E_{x0}}{\eta^*}e^{+j\beta z}\mathbf{a}_y = \frac{E_{x0}}{|\eta|}e^{j\theta}e^{+j\beta z}\mathbf{a}_y$$

where E_{x0} has been assumed real. Eq. (77) applies to any sinusoidal electromagnetic wave and gives both the magnitude and direction of the time-average power density.

D11.6. At frequencies of 1, 100, and 3000 MHz, the dielectric constant of ice made from pure water has values of 4.15, 3.45, and 3.20, respectively, while the loss tangent is 0.12, 0.035, and 0.0009, also respectively. If a uniform plane wave with an amplitude of 100 V/m at $z = 0$ is propagating through such ice, find the time-average power density at $z = 0$ and $z = 10$ m for each frequency.

Ans. 27.1 and 25.7 W/m^2; 24.7 and 6.31 W/m^2; 23.7 and 8.63 W/m^2

11.4 PROPAGATION IN GOOD CONDUCTORS: SKIN EFFECT

As an additional study of propagation with loss, we will investigate the behavior of a *good conductor* when a uniform plane wave is established in it. Such a material satisfies the general high-loss criterion, in which the loss tangent, $\epsilon''/\epsilon' \gg 1$. Applying this to a good conductor leads to the more specific criterion, $\sigma/(\omega\epsilon') \gg 1$. As before, we have an interest in losses that occur on wave transmission *into* a good conductor, and we will find new approximations for the phase constant, attenuation coefficient, and intrinsic impedance. New to us, however, is a modification of the basic problem, appropriate for good conductors. This concerns waves associated with electromagnetic fields existing in an external dielectric that adjoins the conductor surface; in this case, the waves propagate *along* the surface. That portion of the overall field that exists within the conductor will suffer dissipative loss arising from the conduction currents it generates. The overall field therefore attenuates with increasing distance of travel along the surface. This is the mechanism for the resistive transmission line loss that we studied in Chapter 10, and which is embodied in the line resistance parameter, R.

As implied, a good conductor has a high conductivity and large conduction currents. The energy represented by the wave traveling through the material therefore decreases as the wave propagates because ohmic losses are continuously present. When we discussed the loss tangent, we saw that the ratio of conduction current density to the displacement current density in a conducting material is given by $\sigma/\omega\epsilon'$. Choosing a poor metallic conductor and a very high frequency as a conservative example, this ratio[5] for nichrome ($\sigma \doteq 10^6$) at 100 MHz is about 2×10^8. We therefore have a situation where $\sigma/\omega\epsilon' \gg 1$, and we should be able to make several very good approximations to find α, β, and η for a good conductor.

The general expression for the propagation constant is, from (59),

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\sigma}{\omega\epsilon'}}$$

which we immediately simplify to obtain

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{-j\frac{\sigma}{\omega\epsilon'}}$$

or

$$jk = j\sqrt{-j\omega\mu\sigma}$$

But

$$-j = 1\ \angle{-90^\circ}$$

and

$$\sqrt{1\ \angle{-90^\circ}} = 1\ \angle{-45^\circ} = \frac{1}{\sqrt{2}}(1 - j)$$

Therefore

$$jk = j(1 - j)\sqrt{\frac{\omega\mu\sigma}{2}} = (1 + j)\sqrt{\pi f\mu\sigma} = \alpha + j\beta \tag{78}$$

Hence

$$\alpha = \beta = \sqrt{\pi f\mu\sigma} \tag{79}$$

Regardless of the parameters μ and σ of the conductor or of the frequency of the applied field, α and β are equal. If we again assume only an E_x component traveling in the $+z$ direction, then

$$E_x = E_{x0}e^{-z\sqrt{\pi f\mu\sigma}}\cos\left(\omega t - z\sqrt{\pi f\mu\sigma}\right) \tag{80}$$

[5] It is customary to take $\epsilon' = \epsilon_0$ for metallic conductors.

We may tie this field in the conductor to an external field at the conductor surface. We let the region $z > 0$ be the good conductor and the region $z < 0$ be a perfect dielectric. At the boundary surface $z = 0$, (80) becomes

$$E_x = E_{x0} \cos \omega t \qquad (z = 0)$$

This we shall consider as the source field that establishes the fields within the conductor. Since displacement current is negligible,

$$\mathbf{J} = \sigma \mathbf{E}$$

Thus, the conduction current density at any point within the conductor is directly related to **E**:

$$J_x = \sigma E_x = \sigma E_{x0} e^{-z\sqrt{\pi f \mu \sigma}} \cos\left(\omega t - z\sqrt{\pi f \mu \sigma}\right) \tag{81}$$

Equations (80) and (81) contain a wealth of information. Considering first the negative exponential term, we find an exponential decrease in the conduction current density and electric field intensity with penetration into the conductor (away from the source). The exponential factor is unity at $z = 0$ and decreases to $e^{-1} = 0.368$ when

$$z = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

This distance is denoted by δ and is termed the *depth of penetration*, or the *skin depth*,

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\alpha} = \frac{1}{\beta} \tag{82}$$

It is an important parameter in describing conductor behavior in electromagnetic fields. To get some idea of the magnitude of the skin depth, let us consider copper, $\sigma = 5.8 \times 10^7$ S/m, at several different frequencies. We have

$$\delta_{\text{Cu}} = \frac{0.066}{\sqrt{f}}$$

At a power frequency of 60 Hz, $\delta_{\text{Cu}} = 8.53$ mm. Remembering that the power density carries an exponential term $e^{-2\alpha z}$, we see that the power density is multiplied by a factor of $0.368^2 = 0.135$ for every 8.53 mm of distance into the copper.

At a microwave frequency of 10,000 MHz, δ is 6.61×10^{-4} mm. Stated more generally, all fields in a good conductor such as copper are essentially zero at distances greater than a few skin depths from the surface. Any current density or electric field intensity established at the surface of a good conductor decays rapidly as we progress into the conductor. Electromagnetic energy is not transmitted in the interior of a conductor; it travels in the region surrounding the conductor, while the conductor merely guides the waves. We will consider guided propagation in more detail in Chapter 13.

Suppose we have a copper bus bar in the substation of an electric utility company which we wish to have carry large currents, and we therefore select dimensions of 2 by 4 inches. Then much of the copper is wasted, for the fields are greatly reduced in one skin depth, about 8.5 mm.[6] A hollow conductor with a wall thickness of about 12 mm would be a much better design. Although we are applying the results of an analysis for an infinite planar conductor to one of finite dimensions, the fields are attenuated in the finite-size conductor in a similar (but not identical) fashion.

The extremely short skin depth at microwave frequencies shows that only the surface coating of the guiding conductor is important. A piece of glass with an evaporated silver surface 3 μm thick is an excellent conductor at these frequencies.

Next, let us determine expressions for the velocity and wavelength within a good conductor. From (82), we already have

$$\alpha = \beta = \frac{1}{\delta} = \sqrt{\pi f \mu \sigma}$$

Then, as

$$\beta = \frac{2\pi}{\lambda}$$

we find the wavelength to be

$$\lambda = 2\pi\delta \tag{83}$$

Also, recalling that

$$v_p = \frac{\omega}{\beta}$$

we have

$$\boxed{v_p = \omega\delta} \tag{84}$$

For copper at 60 Hz, $\lambda = 5.36$ cm and $v_p = 3.22$ m/s, or about 7.2 mi/h! A lot of us can run faster than that. In free space, of course, a 60 Hz wave has a wavelength of 3100 mi and travels at the velocity of light.

EXAMPLE 11.6

Let us again consider wave propagation in water, but this time we will consider seawater. The primary difference between seawater and fresh water is of course the salt content. Sodium chloride dissociates in water to form Na^+ and Cl^- ions, which, being charged, will move when forced by an electric field. Seawater is thus conductive, and so it will attenuate electromagnetic waves by this mechanism. At frequencies in the vicinity of 10^7 Hz and below, the bound charge effects in water discussed earlier are negligible, and losses in seawater arise principally from the salt-associated conductivity. We consider an incident wave of frequency 1 MHz. We wish to find the skin depth, wavelength, and phase velocity. In seawater, $\sigma = 4$ S/m, and $\epsilon'_r = 81$.

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[6] This utility company operates at 60 Hz.

Solution. We first evaluate the loss tangent, using the given data:

$$\frac{\sigma}{\omega\epsilon'} = \frac{4}{(2\pi \times 10^6)(81)(8.85 \times 10^{-12})} = 8.9 \times 10^2 \gg 1$$

Seawater is therefore a good conductor at 1 MHz (and at frequencies lower than this). The skin depth is

$$\delta = \frac{1}{\sqrt{\pi f \mu\sigma}} = \frac{1}{\sqrt{(\pi \times 10^6)(4\pi \times 10^{-7})(4)}} = 0.25 \text{ m} = 25 \text{ cm}$$

Now

$$\lambda = 2\pi\delta = 1.6 \text{ m}$$

and

$$\nu_p = \omega\delta = (2\pi \times 10^6)(0.25) = 1.6 \times 10^6 \text{ m/sec}$$

In free space, these values would have been $\lambda = 300$ m and of course $\nu = c$.

With a 25-cm skin depth, it is obvious that radio frequency communication in seawater is quite impractical. Notice, however, that δ varies as $1/\sqrt{f}$, so that things will improve at lower frequencies. For example, if we use a frequency of 10 Hz (in the ELF, or extremely low frequency range), the skin depth is increased over that at 1 MHz by a factor of $\sqrt{10^6/10}$, so that

$$\delta(10 \text{ Hz}) \doteq 80 \text{ m}$$

The corresponding wavelength is $\lambda = 2\pi\delta \doteq 500$ m. Frequencies in the ELF range were used for many years in submarine communications. Signals were transmitted from gigantic ground-based antennas (required because the free-space wavelength associated with 10 Hz is 3×10^7 m). The signals were then received by submarines, from which a suspended wire antenna of length shorter than 500 m is sufficient. The drawback is that signal data rates at ELF are slow enough that a single word can take several minutes to transmit. Typically, ELF signals would be used to tell the submarine to initiate emergency procedures, or to come near the surface in order to receive a more detailed message via satellite.

We next turn our attention to finding the magnetic field, H_y, associated with E_x. To do so, we need an expression for the intrinsic impedance of a good conductor. We begin with Eq. (48), Section 11.2, with $\epsilon'' = \sigma/\omega$,

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon'}}$$

Since $\sigma \gg \omega\epsilon'$, we have

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}}$$

which may be written as

$$\eta = \frac{\sqrt{2}\angle 45^\circ}{\sigma\delta} = \frac{(1+j)}{\sigma\delta} \tag{85}$$

Thus, if we write (80) in terms of the skin depth,

$$E_x = E_{x0}e^{-z/\delta}\cos\left(\omega t - \frac{z}{\delta}\right) \tag{86}$$

then

$$H_y = \frac{\sigma\delta E_{x0}}{\sqrt{2}}e^{-z/\delta}\cos\left(\omega t - \frac{z}{\delta} - \frac{\pi}{4}\right) \tag{87}$$

and we see that the maximum amplitude of the magnetic field intensity occurs one-eighth of a cycle later than the maximum amplitude of the electric field intensity at every point.

From (86) and (87) we may obtain the time-average Poynting vector by applying (77),

$$\langle S_z\rangle = \frac{1}{2}\frac{\sigma\delta E_{x0}^2}{\sqrt{2}}e^{-2z/\delta}\cos\left(\frac{\pi}{4}\right)$$

or

$$\langle S_z\rangle = \frac{1}{4}\sigma\delta E_{x0}^2 e^{-2z/\delta}$$

We again note that in a distance of one skin depth the power density is only $e^{-2} = 0.135$ of its value at the surface.

The total average power loss in a width $0 < y < b$ and length $0 < x < L$ in the direction of the current, as shown in Figure 11.3, is obtained by finding the power

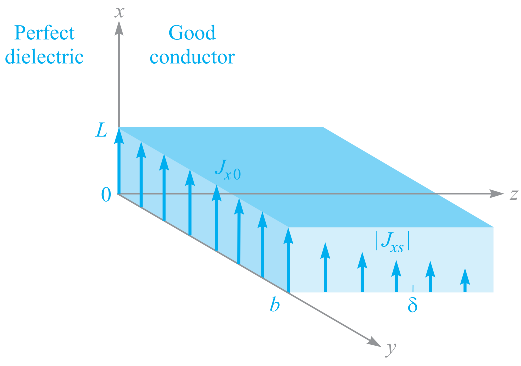

Figure 11.3 The current density $J_x = J_{x0}e^{-z/\delta}e^{-jz/\delta}$ decreases in magnitude as the wave propagates into the conductor. The average power loss in the region $0 < x < L$, $0 < y < b$, $z > 0$, is $\delta bLJ_{x0}^2/4\sigma$ watts.

crossing the conductor surface within this area,

$$P_L = \int_{\text{area}} \langle S_z \rangle da = \int_0^b \int_0^L \frac{1}{4}\sigma\delta E_{x0}^2 e^{-2z/\delta}\Big|_{z=0} dx\, dy = \frac{1}{4}\sigma\delta bLE_{x0}^2$$

In terms of the current density J_{x0} at the surface,

$$J_{x0} = \sigma E_{x0}$$

we have

$$P_L = \frac{1}{4\sigma}\delta bLJ_{x0}^2 \tag{88}$$

Now let us see what power loss would result if the *total* current in a width b were distributed *uniformly* in one skin depth. To find the total current, we integrate the current density over the infinite depth of the conductor,

$$I = \int_0^\infty \int_0^b J_x\, dy\, dz$$

where

$$J_x = J_{x0}e^{-z/\delta}\cos\left(\omega t - \frac{z}{\delta}\right)$$

or in complex exponential notation to simplify the integration,

$$\begin{aligned} J_{xs} &= J_{x0}e^{-z/\delta}e^{-jz/\delta} \\ &= J_{x0}e^{-(1+j)z/\delta} \end{aligned}$$

Therefore,

$$\begin{aligned} I_s &= \int_0^\infty \int_0^b J_{x0}e^{-(1+j)z/\delta} dy\, dz \\ &= J_{x0}be^{-(1+j)z/\delta}\frac{-\delta}{1+j}\Big|_0^\infty \\ &= \frac{J_{x0}b\delta}{1+j} \end{aligned}$$

and

$$I = \frac{J_{x0}b\delta}{\sqrt{2}}\cos\left(\omega t - \frac{\pi}{4}\right)$$

If this current is distributed with a uniform density J' throughout the cross section $0 < y < b$, $0 < z < \delta$, then

$$J' = \frac{J_{x0}}{\sqrt{2}}\cos\left(\omega t - \frac{\pi}{4}\right)$$

The ohmic power loss per unit volume is $\mathbf{J}\cdot\mathbf{E}$, and thus the total instantaneous power dissipated in the volume under consideration is

$$P_{Li}(t) = \frac{1}{\sigma}(J')^2 bL\delta = \frac{J_{x0}^2}{2\sigma}bL\delta\cos^2\left(\omega t - \frac{\pi}{4}\right)$$

The time-average power loss is easily obtained, since the average value of the cosine-squared factor is one-half,

$$P_L = \frac{1}{4\sigma} J_{x0}^2 bL\delta \tag{89}$$

Comparing (88) and (89), we see that they are identical. Thus the average power loss in a conductor with skin effect present may be calculated by assuming that the total current is distributed uniformly in one skin depth. In terms of resistance, we may say that the resistance of a width b and length L of an infinitely thick slab with skin effect is the same as the resistance of a rectangular slab of width b, length L, and thickness δ without skin effect, or with uniform current distribution.

We may apply this to a conductor of circular cross section with little error, provided that the radius a is much greater than the skin depth. The resistance at a high frequency where there is a well-developed skin effect is therefore found by considering a slab of width equal to the circumference $2\pi a$ and thickness δ. Hence

$$R = \frac{L}{\sigma S} = \frac{L}{2\pi a \sigma \delta} \tag{90}$$

A round copper wire of 1 mm radius and 1 km length has a resistance at direct current of

$$R_{dc} = \frac{10^3}{\pi 10^{-6}(5.8 \times 10^7)} = 5.48\ \Omega$$

At 1 MHz, the skin depth is 0.066 mm. Thus $\delta \ll a$, and the resistance at 1 MHz is found by (90),

$$R = \frac{10^3}{2\pi 10^{-3}(5.8 \times 10^7)(0.066 \times 10^{-3})} = 41.5\ \Omega$$

D11.7. A steel pipe is constructed of a material for which $\mu_r = 180$ and $\sigma = 4 \times 10^6$ S/m. The two radii are 5 and 7 mm, and the length is 75 m. If the total current $I(t)$ carried by the pipe is $8 \cos \omega t$ A, where $\omega = 1200\pi$ rad/s, find: (a) the skin depth; (b) the effective resistance; (c) the dc resistance; (d) the time-average power loss.

Ans. 0.766 mm; 0.557 Ω; 0.249 Ω; 17.82 W

11.5 WAVE POLARIZATION

Animations

In the previous sections, we have treated uniform plane waves in which the electric and magnetic field vectors were assumed to lie in fixed directions. Specifically, with the wave propagating along the z axis, **E** was taken to lie along x, which then required **H** to lie along y. This orthogonal relationship between **E**, **H**, and **S** is always true for a uniform plane wave. The directions of **E** and **H** within the plane perpendicular to $\mathbf{a}_z$

may change, however, as functions of time and position, depending on how the wave was generated or on what type of medium it is propagating through. Thus a complete description of an electromagnetic wave would not only include parameters such as its wavelength, phase velocity, and power, but also a statement of the instantaneous orientation of its field vectors. We define the *wave polarization* as the time-dependent electric field vector orientation at a fixed point in space. A more complete characterization of a wave's polarization would in fact include specifying the field orientation at *all* points because some waves demonstrate spatial variations in their polarization. Specifying only the electric field direction is sufficient, since magnetic field is readily found from **E** using Maxwell's equations.

In the waves we have previously studied, **E** was in a fixed straight orientation for all times and positions. Such a wave is said to be *linearly polarized*. We have taken **E** to lie along the x axis, but the field could be oriented in any fixed direction in the xy plane and be linearly polarized. For positive z propagation, the wave would in general have its electric field phasor expressed as

$$\mathbf{E}_s = (E_{x0}\mathbf{a}_x + E_{y0}\mathbf{a}_y)e^{-\alpha z}e^{-j\beta z} \tag{91}$$

where E_{x0} and E_{y0} are constant amplitudes along x and y. The magnetic field is readily found by determining its x and y components directly from those of $\mathbf{E}_s$. Specifically, $\mathbf{H}_s$ for the wave of Eq. (91) is

$$\mathbf{H}_s = [H_{x0}\mathbf{a}_x + H_{y0}\mathbf{a}_y]\, e^{-\alpha z}e^{-j\beta z} = \left[-\frac{E_{y0}}{\eta}\mathbf{a}_x + \frac{E_{x0}}{\eta}\mathbf{a}_y\right] e^{-\alpha z}e^{-j\beta z} \tag{92}$$

The two fields are sketched in Figure 11.4. The figure demonstrates the reason for the minus sign in the term involving E_{y0} in Eq. (92). The direction of power flow, given by $\mathbf{E} \times \mathbf{H}$, is in the positive z direction in this case. A component of **E** in the

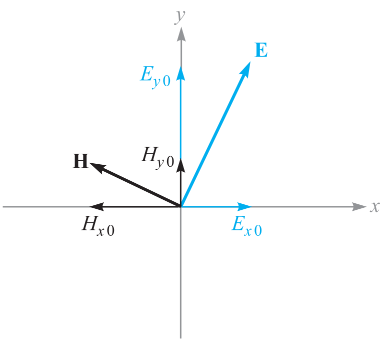

Figure 11.4 Electric and magnetic field configuration for a general linearly polarized plane wave propagating in the forward z direction (out of the page). Field components correspond to those in Eqs. (91) and (92).

positive y direction would require a component of **H** in the negative x direction—thus the minus sign. Using (91) and (92), the power density in the wave is found using (77):

$$\langle \mathbf{S}_z \rangle = \frac{1}{2}\text{Re}\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{1}{2}\text{Re}\{E_{x0}H_{y0}^*(\mathbf{a}_x \times \mathbf{a}_y) + E_{y0}H_{x0}^*(\mathbf{a}_y \times \mathbf{a}_x)\}e^{-2\alpha z}$$
$$= \frac{1}{2}\text{Re}\left\{\frac{E_{x0}E_{x0}^*}{\eta^*} + \frac{E_{y0}E_{y0}^*}{\eta^*}\right\} e^{-2\alpha z}\mathbf{a}_z$$
$$= \frac{1}{2}\text{Re}\left\{\frac{1}{\eta^*}\right\}(|E_{x0}|^2 + |E_{y0}|^2)e^{-2\alpha z}\mathbf{a}_z \ \text{W/m}^2$$

This result demonstrates the idea that our linearly polarized plane wave can be considered as two distinct plane waves having x and y polarizations, whose electric fields are combining *in phase* to produce the total **E**. The same is true for the magnetic field components. This is a critical point in understanding wave polarization, in that *any polarization state can be described in terms of mutually perpendicular components of the electric field and their relative phasing*.

We next consider the effect of a phase difference, ϕ, between E_{x0} and E_{y0}, where $\phi < \pi/2$. For simplicity, we will consider propagation in a lossless medium. The total field in phasor form is

$$\mathbf{E}_s = (E_{x0}\mathbf{a}_x + E_{y0}e^{j\phi}\mathbf{a}_y)e^{-j\beta z} \tag{93}$$

Again, to aid in visualization, we convert this wave to real instantaneous form by multiplying by $e^{j\omega t}$ and taking the real part:

$$\mathbf{E}(z, t) = E_{x0}\cos(\omega t - \beta z)\mathbf{a}_x + E_{y0}\cos(\omega t - \beta z + \phi)\mathbf{a}_y \tag{94}$$

where we have assumed that E_{x0} and E_{y0} are real. Suppose we set $t = 0$, in which case (94) becomes [using $\cos(-x) = \cos(x)$]

$$\mathbf{E}(z, 0) = E_{x0}\cos(\beta z)\mathbf{a}_x + E_{y0}\cos(\beta z - \phi)\mathbf{a}_y \tag{95}$$

The component magnitudes of $\mathbf{E}(z, 0)$ are plotted as functions of z in Figure 11.5. Since time is fixed at zero, the wave is frozen in position. An observer can move along the z axis, measuring the component magnitudes and thus the orientation of the total electric field at each point. Let's consider a crest of E_x, indicated as point a in Figure 11.5. If ϕ were zero, E_y would have a crest at the same location. Since ϕ is not zero (and positive), the crest of E_y that would otherwise occur at point a is now displaced to point b farther down z. The two points are separated by distance ϕ/β. E_y thus *lags behind* E_x when we consider the *spatial* dimension.

Now suppose the observer stops at some location on the z axis, and time is allowed to move forward. Both fields now move in the positive z direction, as (94) indicates. But point b reaches the observer first, followed by point a. So we see that E_y *leads* E_x when we consider the *time* dimension. In either case (fixed t and varying z, or vice versa) the observer notes that the net field rotates about the z axis while its magnitude changes. Considering a starting point in z and t, at which the field has a given orientation and magnitude, the wave will return to the same orientation and

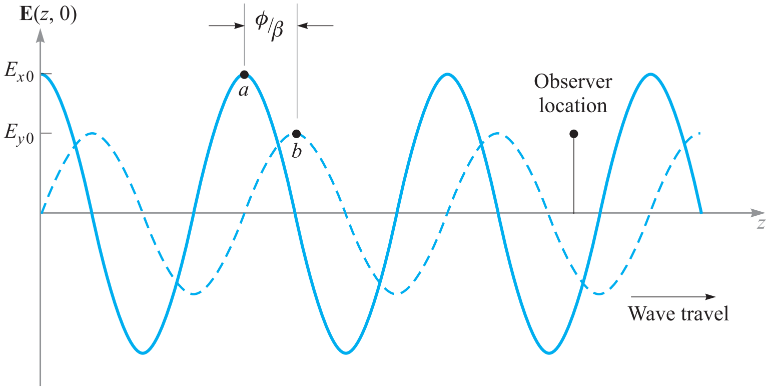

Figure 11.5 Plots of the electric field component magnitudes in Eq. (95) as functions of z. Note that the y component lags behind the x component in z. As time increases from zero, both waves travel to the right, as per Eq. (94). Thus, to an observer at a fixed location, the y component leads in time.

magnitude at a distance of one wavelength in z (for fixed t) or at a time $t = 2\pi/\omega$ later (at a fixed z).

For illustration purposes, if we take the length of the field vector as a measure of its magnitude, we find that at a fixed position, the tip of the vector traces out the shape of an ellipse over time $t = 2\pi/\omega$. The wave is said to be *elliptically polarized*. Elliptical polarization is in fact the most general polarization state of a wave, since it encompasses any magnitude and phase difference between E_x and E_y. Linear polarization is a special case of elliptical polarization in which the phase difference is zero.

Another special case of elliptical polarization occurs when $E_{x0} = E_{y0} = E_0$ and when $\phi = \pm\pi/2$. The wave in this case exhibits *circular polarization*. To see this, we incorporate these restrictions into Eq. (94) to obtain

$$\begin{aligned}\mathbf{E}(z,t) &= E_0[\cos(\omega t - \beta z)\mathbf{a}_x + \cos(\omega t - \beta z \pm \pi/2)\mathbf{a}_y] \\ &= E_0[\cos(\omega t - \beta z)\mathbf{a}_x \mp \sin(\omega t - \beta z)\mathbf{a}_y]\end{aligned} \tag{96}$$

If we consider a fixed position along z (such as $z = 0$) and allow time to vary, (96), with $\phi = +\pi/2$, becomes

$$\boxed{\mathbf{E}(0,t) = E_0[\cos(\omega t)\mathbf{a}_x - \sin(\omega t)\mathbf{a}_y]} \tag{97}$$

If we choose $-\pi/2$ in (96), we obtain

$$\boxed{\mathbf{E}(0,t) = E_0[\cos(\omega t)\mathbf{a}_x + \sin(\omega t)\mathbf{a}_y]} \tag{98}$$

The field vector of Eq. (98) rotates in the counterclockwise direction in the xy plane, while maintaining constant amplitude E_0, and so the tip of the vector traces out a circle. Figure 11.6 shows this behavior.

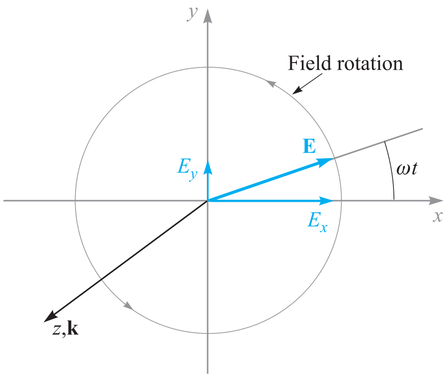

Figure 11.6 Electric field in the xy plane of a right circularly polarized plane wave, as described by Eq. (98). As the wave propagates in the forward z direction, the field vector rotates counterclockwise in the xy plane.

Choosing $+\pi/2$ leads to (97), whose field vector rotates in the clockwise direction. The *handedness* of the circular polarization is associated with the rotation and propagation directions in the following manner: The wave exhibits *left circular polarization* (l.c.p.) if, when orienting the left hand with the thumb in the direction of propagation, the fingers curl in the rotation direction of the field with time. The wave exhibits *right circular polarization* (r.c.p.) if, with the right-hand thumb in the propagation direction, the fingers curl in the field rotation direction.[7] Thus, with forward z propagation, (97) describes a left circularly polarized wave, and (98) describes a right circularly polarized wave. The same convention is applied to elliptical polarization, in which the descriptions *left elliptical polarization* and *right elliptical polarization* are used.

Using (96), the instantaneous angle of the field from the x direction can be found for any position along z through

$$\theta(z,t) = \tan^{-1}\left(\frac{E_y}{E_x}\right) = \tan^{-1}\left(\frac{\mp\sin(\omega t - \beta z)}{\cos(\omega t - \beta z)}\right) = \mp(\omega t - \beta z) \qquad (99)$$

where again the minus sign (yielding l.c.p. for positive z travel) applies for the choice of $\phi = +\pi/2$ in (96); the plus sign (yielding r.c.p. for positive z travel) is used if

[7] This convention is reversed by some workers (most notably in optics) who emphasize the importance of the *spatial* field configuration. Note that r.c.p. by our definition is formed by propagating a spatial field that is in the shape of a *left-handed* screw, and for that reason it is sometimes called left circular polarization (see Figure 11.7). Left circular polarization as we define it results from propagating a spatial field in the shape of a right-handed screw, and it is called right circular polarization by the spatial enthusiasts. Caution is obviously necessary in interpreting what is meant when polarization handedness is stated in an unfamiliar text.

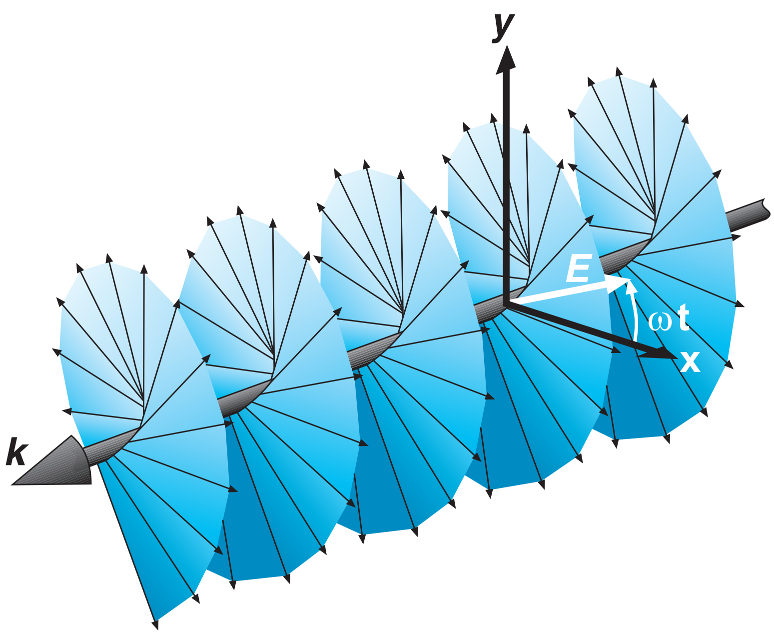

Figure 11.7 Representation of a right circularly polarized wave. The electric field vector (in white) will rotate toward the *y* axis as the entire wave moves through the *xy* plane in the direction of **k**. This counterclockwise rotation (when looking toward the wave source) satisfies the temporal right-handed rotation convention as described in the text. The wave, however, appears as a left-handed screw, and for this reason it is called left circular polarization in the other convention.

$\phi = -\pi/2$. If we choose $z = 0$, the angle becomes simply ωt, which reaches 2π (one complete rotation) at time $t = 2\pi/\omega$. If we choose $t = 0$ and allow z to vary, we form a corkscrew-like field pattern. One way to visualize this is to consider a spiral staircase–shaped pattern, in which the field lines (stairsteps) are perpendicular to the z (or staircase) axis. The relationship between this spatial field pattern and the resulting time behavior at fixed z as the wave propagates is shown in an artist's conception in Figure 11.7.

The handedness of the polarization is changed by reversing the pitch of the corkscrew pattern. The spiral staircase model is only a visualization aid. It must be remembered that the wave is still a uniform plane wave whose fields at any position along z are infinite in extent over the transverse plane.

There are many uses of circularly polarized waves. Perhaps the most obvious advantage is that reception of a wave having circular polarization does not depend on the antenna orientation in the plane normal to the propagation direction. Dipole antennas, for example, are required to be oriented along the electric field direction of the signal they receive. If circularly polarized signals are transmitted, the receiver orientation requirements are relaxed considerably. In optics, circularly polarized light can be passed through a polarizer of any orientation, thus yielding linearly polarized light in any direction (although one loses half the original power this way). Other uses involve treating linearly polarized light as a superposition of circularly polarized waves, to be described next.

Circularly polarized light can be generated using an *anisotropic* medium—a material whose permittivity is a function of electric field direction. Many crystals have this property. A crystal orientation can be found such that along one direction (say, the x axis), the permittivity is lowest, while along the orthogonal direction (y axis), the permittivity is highest. The strategy is to input a linearly polarized wave with its field vector at 45 degrees to the x and y axes of the crystal. It will thus have equal-amplitude x and y components in the crystal, and these will now propagate in the z direction at different speeds. A phase difference (or *retardation*) accumulates between the components as they propagate, which can reach $\pi/2$ if the crystal is long enough. The wave at the output thus becomes circularly polarized. Such a crystal, cut to the right length and used in this manner, is called a *quarter-wave plate,* since it introduces a relative phase shift of $\pi/2$ between E_x and E_y, which is equivalent to $\lambda/4$.

It is useful to express circularly polarized waves in phasor form. To do this, we note that (96) can be expressed as

$$\mathbf{E}(z,t) = \text{Re}\left\{E_0 e^{j\omega t} e^{-j\beta z}\left[\mathbf{a}_x + e^{\pm j\pi/2}\mathbf{a}_y\right]\right\}$$

Using the fact that $e^{\pm j\pi/2} = \pm j$, we identify the phasor form as:

$$\mathbf{E}_s = E_0(\mathbf{a}_x \pm j\mathbf{a}_y)e^{-j\beta z} \tag{100}$$

where the plus sign is used for left circular polarization and the minus sign for right circular polarization. If the wave propagates in the negative z direction, we have

$$\mathbf{E}_s = E_0(\mathbf{a}_x \pm j\mathbf{a}_y)e^{+j\beta z} \tag{101}$$

where in this case the positive sign applies to right circular polarization and the minus sign to left circular polarization. The student is encouraged to verify this.

EXAMPLE 11.7

Let us consider the result of superimposing left and right circularly polarized fields of the same amplitude, frequency, and propagation direction, but where a phase shift of δ radians exists between the two.

Solution. Taking the waves to propagate in the $+z$ direction, and introducing a relative phase, δ, the total phasor field is found, using (100):

$$\mathbf{E}_{sT} = \mathbf{E}_{sR} + \mathbf{E}_{sL} = E_0[\mathbf{a}_x - j\mathbf{a}_y]e^{-j\beta z} + E_0[\mathbf{a}_x + j\mathbf{a}_y]e^{-j\beta z}e^{j\delta}$$

Grouping components together, this becomes

$$\mathbf{E}_{sT} = E_0[(1 + e^{j\delta})\mathbf{a}_x - j(1 - e^{j\delta})\mathbf{a}_y]e^{-j\beta z}$$

Factoring out an overall phase term, $e^{j\delta/2}$, we obtain

$$\mathbf{E}_{sT} = E_0 e^{j\delta/2}\left[(e^{-j\delta/2} + e^{j\delta/2})\mathbf{a}_x - j(e^{-j\delta/2} - e^{j\delta/2})\mathbf{a}_y\right]e^{-j\beta z}$$

From Euler's identity, we find that $e^{j\delta/2} + e^{-j\delta/2} = 2\cos\delta/2$, and $e^{j\delta/2} - e^{-j\delta/2} = 2j\sin\delta/2$. Using these relations, we obtain

$$\mathbf{E}_{sT} = 2E_0[\cos(\delta/2)\mathbf{a}_x + \sin(\delta/2)\mathbf{a}_y]e^{-j(\beta z - \delta/2)} \tag{102}$$

We recognize (102) as the electric field of a *linearly polarized* wave, whose field vector is oriented at angle $\delta/2$ from the x axis.

Example 11.7 shows that any linearly polarized wave can be expressed as the sum of two circularly polarized waves of opposite handedness, where the linear polarization direction is determined by the relative phase difference between the two waves. Such a representation is convenient (and necessary) when considering, for example, the propagation of linearly polarized light through media which contain organic molecules. These often exhibit spiral structures having left- or right-handed pitch, and they will thus interact differently with left- or right-hand circular polarization. As a result, the left circular component can propagate at a different speed than the right circular component, and so the two waves will accumulate a phase difference as they propagate. As a result, the direction of the linearly polarized field vector at the output of the material will differ from the direction that it had at the input. The extent of this rotation can be used as a measurement tool to aid in material studies.

Polarization issues will become extremely important when we consider wave reflection in Chapter 12.

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CHAPTER 11 PROBLEMS

WWW Quizzes

11.1 Show that $E_{xs} = Ae^{j(k_0 z + \phi)}$ is a solution of the vector Helmholtz equation, Eq. (30), for $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ and any ϕ and A.

11.2 A 10 GHz uniform plane wave propagates in a lossless medium for which $\epsilon_r = 8$ and $\mu_r = 2$. Find (*a*) v_p; (*b*) β; (*c*) λ; (*d*) $\mathbf{E}_s$; (*e*) $\mathbf{H}_s$; (*f*) $\langle\mathbf{S}\rangle$.

11.3 An **H** field in free space is given as $\mathcal{H}(x,t) = 10\cos(10^8 t - \beta x)\mathbf{a}_y$ A/m. Find (*a*) β; (*b*) λ; (*c*) $\mathcal{E}(x,t)$ at $P(0.1, 0.2, 0.3)$ at $t = 1$ ns.

11.4 Small antennas have low efficiencies (as will be seen in Chapter 14), and the efficiency increases with size up to the point at which a critical dimension of the antenna is an appreciable fraction of a wavelength, say $\lambda/8$. (*a*) An antenna that is 12 cm long is operated in air at 1 MHz. What fraction of a wavelength long is it? (*b*) The same antenna is embedded in a ferrite material for which $\epsilon_r = 20$ and $\mu_r = 2,000$. What fraction of a wavelength is it now?

11.5 A 150 MHz uniform plane wave in free space is described by $\mathbf{H}_s = (4 + j10)(2\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z}$ A/m. (*a*) Find numerical values for ω, λ, and β. (*b*) Find $\mathcal{H}(z, t)$ at $t = 1.5$ ns, $z = 20$ cm. (*c*) What is $|E|_{\max}$?

11.6 A uniform plane wave has electric field $\mathbf{E}_s = (E_{y0}\,\mathbf{a}_y - E_{z0}\,\mathbf{a}_z)\,e^{-\alpha x}\,e^{-j\beta x}$ V/m. The intrinsic impedance of the medium is given as $\eta = |\eta|\,e^{j\phi}$, where ϕ is a constant phase. (*a*) Describe the wave polarization and state the direction of propagation. (*b*) Find $\mathbf{H}_s$. (*c*) Find $\mathcal{E}(x, t)$ and $\mathcal{H}(x, t)$. (*d*) Find $< \mathbf{S} >$ in W/m^2. (*e*) Find the time-average power in watts that is intercepted by an antenna of rectangular cross-section, having width w and height h, suspended parallel to the yz plane, and at a distance d from the wave source.

11.7 The phasor magnetic field intensity for a 400 MHz uniform plane wave propagating in a certain lossless material is $(2\mathbf{a}_y - j5\mathbf{a}_z)e^{-j25x}$ A/m. Knowing that the maximum amplitude of **E** is 1500 V/m, find β, η, λ, ν_p, ϵ_r, μ_r, and $\mathcal{H}(x, y, z, t)$.

11.8 An electric field in free space is given in spherical coordinates as $\mathbf{E}_s(r) = E_0(r)e^{-jkr}\,\mathbf{a}_\theta$ V/m. (*a*) Find $\mathbf{H}_s(r)$ assuming uniform plane wave behavior. (*b*) Find $< \mathbf{S} >$. (*c*) Express the average outward power in watts through a closed spherical shell of radius r, centered at the origin. (*d*) Establish the required functional form of $E_0(r)$ that will enable the power flow in part *c* to be independent of radius. With this condition met, the given field becomes that of an *isotropic radiator* in a lossless medium (radiating equal power density in all directions).

11.9 A certain lossless material has $\mu_r = 4$ and $\epsilon_r = 9$. A 10-MHz uniform plane wave is propagating in the $\mathbf{a}_y$ direction with $E_{x0} = 400$ V/m and $E_{y0} = E_{z0} = 0$ at $P(0.6, 0.6, 0.6)$ at $t = 60$ ns. Find (*a*) β, λ, ν_p, and η; (*b*) $\mathcal{E}(y, t)$; (*c*) $\mathcal{H}(y, t)$.

11.10 In a medium characterized by intrinsic impedance $\eta = |\eta|e^{j\phi}$, a linearly polarized plane wave propagates, with magnetic field given as $\mathbf{H}_s = (H_{0y}\mathbf{a}_y + H_{0z}\mathbf{a}_z)e^{-\alpha x}e^{-j\beta x}$. Find (*a*) $\mathbf{E}_s$; (*b*) $\mathcal{E}(x, t)$; (*c*) $\mathcal{H}(x, t)$; (*d*) $\langle \mathbf{S} \rangle$.

11.11 A 2 GHz uniform plane wave has an amplitude $E_{y0} = 1.4$ kV/m at $(0, 0, 0, t = 0)$ and is propagating in the $\mathbf{a}_z$ direction in a medium where $\epsilon'' = 1.6 \times 10^{-11}$ F/m, $\epsilon' = 3.0 \times 10^{-11}$ F/m, and $\mu = 2.5\ \mu$H/m. Find (*a*) E_y at $P(0, 0, 1.8$ cm$)$ at 0.2 ns; (*b*) H_x at P at 0.2 ns.

11.12 Describe how the attenuation coefficient of a liquid medium, assumed to be a good conductor, could be determined through measurement of wavelength in the liquid at a known frequency. What restrictions apply? Could this method be used to find the conductivity as well?

11.13 Let $jk = 0.2 + j1.5$ m^{-1} and $\eta = 450 + j60\,\Omega$ for a uniform plane propagating in the $\mathbf{a}_z$ direction. If $\omega = 300$ Mrad/s, find μ, ϵ', and ϵ'' for the medium.

11.14 A certain nonmagnetic material has the material constants $\epsilon'_r = 2$ and $\epsilon''/\epsilon' = 4 \times 10^{-4}$ at $\omega = 1.5$ Grad/s. Find the distance a uniform plane wave can propagate through the material before (*a*) it is attenuated by 1 Np; (*b*) the power level is reduced by one-half; (*c*) the phase shifts 360°.

11.15 A 10 GHz radar signal may be represented as a uniform plane wave in a sufficiently small region. Calculate the wavelength in centimeters and the attenuation in nepers per meter if the wave is propagating in a nonmagnetic material for which (*a*) $\epsilon'_r = 1$ and $\epsilon''_r = 0$; (*b*) $\epsilon'_r = 1.04$ and $\epsilon''_r = 9.00 \times 10^{-4}$; (*c*) $\epsilon'_r = 2.5$ and $\epsilon''_r = 7.2$.

11.16 Consider the power dissipation term, $\int \mathbf{E} \cdot \mathbf{J} dv$, in Poynting's theorem (Eq. (70)). This gives the power lost to heat within a volume into which electromagnetic waves enter. The term $p_d = \mathbf{E} \cdot \mathbf{J}$ is thus the power dissipation per unit volume in W/m^3. Following the same reasoning that resulted in Eq. (77), the time-average power dissipation per volume will be $< p_d >= (1/2)\mathcal{R}e\left\{\mathbf{E}_s \cdot \mathbf{J}_s^*\right\}$. (a) Show that in a conducting medium, through which a uniform plane wave of amplitude E_0 propagates in the forward z direction, $< p_d >= (\sigma/2)|E_0|^2 e^{-2\alpha z}$. (b) Confirm this result for the special case of a good conductor by using the left hand side of Eq. (70), and consider a very small volume.

11.17 Let $\eta = 250 + j30\,\Omega$ and $jk = 0.2 + j2\text{m}^{-1}$ for a uniform plane wave propagating in the $\mathbf{a}_z$ direction in a dielectric having some finite conductivity. If $|E_s| = 400$ V/m at $z = 0$, find (*a*) $\langle \mathbf{S} \rangle$ at $z = 0$ and $z = 60$ cm; (*b*) the average ohmic power dissipation in watts per cubic meter at $z = 60$ cm.

11.18 Given a 100-MHz uniform plane wave in a medium known to be a good dielectric, the phasor electric field is $\mathcal{E}_s = 4e^{-0.5z}e^{-j20z}\mathbf{a}_x$ V/m. Determine (*a*) ϵ'; (*b*) ϵ''; (*c*) η; (*d*) $\mathbf{H}_s$; (*e*) $\langle \mathbf{S} \rangle$; (*f*) the power in watts that is incident on a rectangular surface measuring 20 m × 30 m at $z = 10$ m.

11.19 Perfectly conducting cylinders with radii of 8 mm and 20 mm are coaxial. The region between the cylinders is filled with a perfect dielectric for which $\epsilon = 10^{-9}/4\pi$ F/m and $\mu_r = 1$. If $\mathcal{E}$ in this region is $(500/\rho)\cos(\omega t - 4z)\mathbf{a}_\rho$ V/m, find (*a*) ω, with the help of Maxwell's equations in cylindrical coordinates; (*b*) $\mathcal{H}(\rho, z, t)$; (*c*) $\langle \mathbf{S}(\rho, z, t) \rangle$; (*d*) the average power passing through every cross section $8 < \rho < 20$ mm, $0 < \phi < 2\pi$.

11.20 Voltage breakdown in air at standard temperature and pressure occurs at an electric field strength of approximately 3×10^6 V/m. This becomes an issue in some high-power optical experiments, in which tight focusing of light may be necessary. Estimate the lightwave power in watts that can be focused into a cylindrical beam of 10μm radius before breakdown occurs. Assume uniform plane wave behavior (although this assumption will produce an answer that is higher than the actual number by as much as a factor of 2, depending on the actual beam shape).

11.21 The cylindrical shell, 1 cm$< \rho <$ 1.2 cm, is composed of a conducting material for which $\sigma = 10^6$ S/m. The external and internal regions are nonconducting. Let $H_\phi = 2000$ A/m at $\rho = 1.2$ cm. Find (*a*) **H** everywhere; (*b*) **E** everywhere; (*c*) $\langle \mathbf{S} \rangle$ everywhere.

11.22 The inner and outer dimensions of a coaxial copper transmission line are 2 and 7 mm, respectively. Both conductors have thicknesses much greater than δ. The dielectric is lossless and the operating frequency is 400 MHz. Calculate the resistance per meter length of the (*a*) inner conductor; (*b*) outer conductor; (*c*) transmission line.

11.23 A hollow tubular conductor is constructed from a type of brass having a conductivity of 1.2×10^7 S/m. The inner and outer radii are 9 and 10 mm, respectively. Calculate the resistance per meter length at a frequency of (*a*) dc; (*b*) 20 MHz; (*c*) 2 GHz.

11.24 (*a*) Most microwave ovens operate at 2.45 GHz. Assume that $\sigma = 1.2 \times 10^6$ S/m and $\mu_r = 500$ for the stainless steel interior, and find the depth of penetration. (*b*) Let $E_s = 50\angle 0^\circ$ V/m at the surface of the conductor, and plot a curve of the amplitude of E_s versus the angle of E_s as the field propagates into the stainless steel.

11.25 A good conductor is planar in form, and it carries a uniform plane wave that has a wavelength of 0.3 mm and a velocity of 3×10^5 m/s. Assuming the conductor is nonmagnetic, determine the frequency and the conductivity.

11.26 The dimensions of a certain coaxial transmission line are $a = 0.8$ mm and $b = 4$ mm. The outer conductor thickness is 0.6 mm, and all conductors have $\sigma = 1.6 \times 10^7$ S/m. (*a*) Find R, the resistance per unit length at an operating frequency of 2.4 GHz. (*b*) Use information from Sections 6.3 and 8.10 to find C and L, the capacitance and inductance per unit length, respectively. The coax is air-filled. (*c*) Find α and β if $\alpha + j\beta = \sqrt{j\omega C(R + j\omega L)}$.

11.27 The planar surface $z = 0$ is a brass-Teflon interface. Use data available in Appendix C to evaluate the following ratios for a uniform plane wave having $\omega = 4 \times 10^{10}$ rad/s: (*a*) $\alpha_{\text{Tef}}/\alpha_{\text{brass}}$; (*b*) $\lambda_{\text{Tef}}/\lambda_{\text{brass}}$; (*c*) $v_{\text{Tef}}/v_{\text{brass}}$.

11.28 A uniform plane wave in free space has electric field vector given by $\mathbf{E}_s = 10e^{-j\beta x}\mathbf{a}_z + 15e^{-j\beta x}\mathbf{a}_y$ V/m. (*a*) Describe the wave polarization. (*b*) Find $\mathbf{H}_s$. (*c*) Determine the average power density in the wave in W/m^2.

11.29 Consider a left circularly polarized wave in free space that propagates in the forward z direction. The electric field is given by the appropriate form of Eq. (100). Determine (*a*) the magnetic field phasor, $\mathbf{H}_s$; (*b*) an expression for the average power density in the wave in W/m^2 by direct application of Eq. (77).

11.30 In an *anisotropic* medium, permittivity varies with electric field *direction*, and is a property seen in most crystals. Consider a uniform plane wave propagating in the z direction in such a medium, and which enters the material with equal field components along the x and y axes. The field phasor will take the form:

$$\mathbf{E}_s(z) = E_0(\mathbf{a}_x + \mathbf{a}_y\, e^{j\Delta\beta z})\, e^{-j\beta z}$$

where $\Delta\beta = \beta_x - \beta_y$ is the difference in phase constants for waves that are linearly polarized in the x and y directions. Find distances into the material (in terms of $\Delta\beta$) at which the field is (*a*) linearly polarized and (*b*) circularly polarized. (*c*) Assume intrinsic impedance η that is approximately constant with field orientation and find $\mathbf{H}_s$ and $< \mathbf{S} >$.

11.31 A linearly polarized uniform plane wave, propagating in the forward z direction, is input to a lossless anisotropic material, in which the dielectric constant encountered by waves polarized along $y(\epsilon_{ry})$ differs from that seen by waves polarized along $x(\epsilon_{rx})$. Suppose $\epsilon_{rx} = 2.15$, $\epsilon_{ry} = 2.10$, and the wave electric field at input is polarized at 45° to the positive x and y axes. (*a*) Determine, in terms of the free space wavelength, λ, the shortest length of the material, such that the wave, as it emerges from the output, is circularly polarized. (*b*) Will the output wave be right or left circularly polarized? Problem 11.30 is good background.

11.32 Suppose that the length of the medium of Problem 11.31 is made to be *twice* that determined in the problem. Describe the polarization of the output wave in this case.

11.33 Given a wave for which $\mathbf{E}_s = 15e^{-j\beta z}\mathbf{a}_x + 18e^{-j\beta z}e^{j\phi}\mathbf{a}_y$ V/m in a medium characterized by complex intrinsic impedance, η (*a*) find $\mathbf{H}_s$; (*b*) determine the average power density in W/m^2.

11.34 Given a general elliptically polarized wave as per Eq. (93):

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{j\phi}\mathbf{a}_y]e^{-j\beta z}$$

(*a*) Show, using methods similar to those of Example 11.7, that a linearly polarized wave results when superimposing the given field and a phase-shifted field of the form:

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{-j\phi}\mathbf{a}_y]e^{-j\beta z}e^{j\delta}$$

where δ is a constant. (*b*) Find δ in terms of ϕ such that the resultant wave is linearly polarized along x.